Spectral sequences

Spectral sequences are a powerful computational tool in topology. Computing with spectral sequences is a bit like computing integrals in calculus; it is helpful to have ingenuity & a bag of tricks – & even that may not be enough!

Filtered complexes

Let $C_\bullet$ be a chain complex, & $F_0C_\bullet$ a sub-complex. Then we have a short exact sequence

$$0 \to F_0C_\bullet \to C_\bullet \to C_\bullet/F_0C_\bullet \to 0$$

This gives rise to a long exact sequence in homology

$$\cdots \to H_i(F_0C_\bullet) \to H_i(C_\bullet) \to H_i(C_\bullet/F_0C_\bullet) \xrightarrow{\delta} H_{i-1}(F_0C_\bullet) \to \cdots$$

Suppose we know $H_*(F_0C_\bullet)$ & $H_*(C_\bullet/F_0C_\bullet)$. Can we compute $H_*(C_\bullet)$? The long exact sequence gives rise to SES's

$$0 \to \operatorname{coker}(\delta) \to H_*(C_\bullet) \to \ker(\delta) \to 0$$

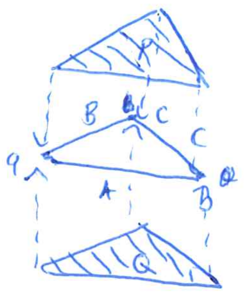
So, a procedure for computing $H_*(C_\bullet)$ is the following:

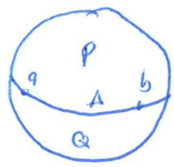
1. Compute $H_*(F_0C_\bullet)$ & $H_*(C_\bullet/F_0C_\bullet)$
2. Consider the two-term chain complex
$$H_*(C_\bullet/F_0C_\bullet) \xrightarrow{\delta} H_*(F_0C_\bullet)$$
Denote its homology groups by G_1H_* & G_0H_*.
3. There is a short exact sequence
$$0 \to G_0H_* \to H_*(C_\bullet) \to G_1H_* \to 0$$

This determines $H_*(C_\bullet)$ up to extension.

Question: What if we have a filtered space
$\cdots \subset F_pX \subset F_{p+1}X \subset \cdots$

Example: Consider the following model of S^2





The associated chain complex is:

$$0 \longrightarrow \mathbb{Z}\{P, Q\} \xrightarrow{d} \mathbb{Z}\{A, B, C\} \xrightarrow{d} \mathbb{Z}\{a, b, c\} \longrightarrow 0$$

$d(P) = C - B + A \qquad d(A) = b - a$

$d(Q) = C - B + A \qquad d(B) = c - a$

$d(C) = c - b$

Instead of directly computing the homology, we consider the following filtration.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}\{P, Q\} & \xrightarrow{d} & \mathbb{Z}\{A, B, C\} & \xrightarrow{d} & \mathbb{Z}\{a, b, c\} & \longrightarrow & 0 \\
 & & & & \cup\!\mid & & & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}\{A, B\} & \longrightarrow & \mathbb{Z}\{a, b, c\} & \longrightarrow & 0 \\
 & & & & \cup\!\mid & & & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}\{A\} & \longrightarrow & \mathbb{Z}\{a, b\} & \longrightarrow & 0
\end{array}$$

Each row is a chain complex, & we have inclusions as indicated. We can therefore consider the quotient of successive rows. We call this quotient $E^0_{\mathbb{Z}}$ & it inherits a differential from d, called d_0.

$$\begin{array}{cccccccccl}
0 & \longrightarrow & \mathbb{Z}\{P, Q\} & \xrightarrow{d_0} & \mathbb{Z}\{C\} & \longrightarrow & 0 & \longrightarrow & 0 & d_0(P) = C, \ d_0(Q) = C \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}\{B\} & \xrightarrow{d_0} & \mathbb{Z}\{c\} & \longrightarrow & 0 & d_0(B) = c \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}\{A\} & \xrightarrow{d_0} & \mathbb{Z}\{a, b\} & \longrightarrow & 0 & d_0(A) = b - a
\end{array}$$

The homology is

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}\{P - Q\} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}\{a\} & \longrightarrow & 0
\end{array}$$

This is exactly the homology of S^2!

Def: A filtered R-module is an R-module A together with an increasing sequence of submodules $F_p A \subset F_{p+1} A$ indexed by $p \in \mathbb{Z}$ such that $\bigcup_p F_p A = A$ & $\bigcap F_p A = \{0\}$. The filtration is <u>bounded</u> if $F_p A = \{0\}$ for p sufficiently small, & $F_p A = A$ for p sufficiently large. The associated graded module is defined by $G_p A = F_p A / F_{p-1} A$, ie,

$$0 \to F_{p-1} A \to F_p A \to G_p A \to 0.$$

Def: A filtered chain complex is a chain complex (C_*, ∂) together with a filtration $\{F_p C_i\}$ of each C_i such that the differential preserves the filtration $\partial(F_p C_i) \subset F_p C_{i-1}$. Then, ∂ induces $\partial: G_p C_i \to G_p C_{i-1}$.

Rem: The filtration on C_* induces a filtration on the homology of C_*, defined by

$$F_p H_i(C_*) = \{ \alpha \in H_i(C_*) \mid (\exists x \in F_p(C_i),\ \alpha = [x]) \}$$

This has associated graded pieces $G_p H_i(C_*)$.

Idea: Suppose we want to compute $H_*(C_*)$, & that we can compute $H_*(G_p(C_*))$. Does $H_*(G_p(C_*))$ determine $G_p H_*(C_*)$?

Rem: If $F_{-1} C_* = \{0\}$ & $F_1 C_* = C_*$, then $G_p H_*(C_*)$ is the homology of the two-term chain complex

$$H_*(G_1(C_*)) \xrightarrow{\partial} H_*(G_0(C_*)).$$

When the filtration has more terms, the homology can be computed by "successive approximations" as we now explain.

The homology of a filtered chain complex

Let $(F_p C_*, \partial)$ be a filtered chain complex. Let us write

$$E^0_{p,q} = G_p C_{p+q} = \frac{F_p C_{p+q}}{F_{p-1} C_{p+q}}$$

The differential ∂ induces a differential on E^0,

$$\partial_0 : E^0_{p,q} \longrightarrow E^0_{p,q-1}$$

We denote the homology of the associated graded by

$$E^1_{p,q} = H_{p+q}(G_p C_*).$$

This is a "first order approximation" to $H_*(C_*)$. We now define

$$\partial_1 : E^1_{p,q} \longrightarrow E^1_{p-1,q}$$

as follows: a homology class $E^1_{p,q}$ can be represented by a chain $x \in F_p C_{p+q}$ such that $\partial x \in F_{p-1} C_{p+q-1}$. We define $\partial_1(x) = [\partial x]$. Because $\partial^2 = 0$, we check that $\partial_1^2 = 0$ & ∂_1 is well defined. We now define

$$E^2_{p,q} = \frac{\operatorname{Ker}(\partial_1 : E^1_{p,q} \longrightarrow E^1_{p-1,q})}{\operatorname{Im}(\partial_1 : E^1_{p+1,q} \longrightarrow E^1_{p,q})}.$$

We can define an "r-th" order approximation to $G_p H_{p+q}(C_*)$ by

$$E^r_{p,q} = \frac{\{x \in F_p C_{p+q} \mid \partial x \in F_{p-r} C_{p+q-1}\}}{F_{p-1} C_{p+q} + \partial(F_{p+r-1} C_{p+q+1})}$$

The notation denotes the quotient of the numerator by its intersection with the denominator.

So, instead of considering cycles, we consider chains in F_p whose differential vanishes "to order r", i.e. lies in F_{p-r}, & instead of modding out by the entire image of ∂, we only mod out by $\partial(F_{p+r-1})$.

__Lemma__: Let $(F_p C_*, \partial)$ denote a filtered chain complex, & define $E^r_{p,q}$ as above. Then,

(a) ∂ induces a map

$$\partial_r : E^r_{p,q} \longrightarrow E^r_{p-r,\, q+r-1}$$

satisfying $\partial_r^2 = 0$

(b) E^{r+1} is the homology of the chain complex (E^r, ∂_r), i.e.,

$$E^{r+1}_{p,q} = \frac{\operatorname{Ker}(\partial_r : E^r_{p,q} \to E^r_{p-r,\,q+r-1})}{\operatorname{Im}(\partial_r : E^r_{p+r,\,q-r+1} \to E^r_{p,q})}$$

(c) $E^1_{p,q} = H_{p+q}(G_p(C_*))$.

(d) The filtration of C_i is bounded for each i, then for every p,q if r is large enough

$$E^r_{p,q} = G_p H_{p+q}(C_*).$$

[One writes $E^1_{p,q} = H_{p+q}(G_p(C_*)) \Rightarrow H_{p+q}(C_*)$]

__Proof__: Messy algebraic exercise – the hard part is finding the right statement!

Example: Let $C_*(X)$ denote the singular chains on a CW complex X. Define a filtration by

$$F_p C_*(X) = C_*(X^p),$$

the chains of the p-skeleton. The associated graded is

$$E^0_{p,q} = \frac{C_{p+q}(X^p)}{C_{p+q}(X^{p-1})}.$$

The homology of this is, by definition, the relative homology

$$E^1_{p,q} = H_{p+q}(X^p, X^{p-1}).$$

Recall from last semester that

$$H_{p+q}(X^p, X^{p-1}) \cong \begin{cases} C_p^{cell}(X), & q=0 \\ 0, & q \neq 0, \end{cases}$$

where $C_p^{cell}(X)$ is a free $\mathbb{Z}$-module with one generator for each p-cell. Furthermore, we have a differential $\partial: H_p(X^p, X^{p-1}) \to H_{p-1}(X^{p-1}, X^{p-2})$ induced by stitching together long exact sequences. This is exactly $\partial_1: E^1_{p,0} \to E^1_{p-1,0}$.

Therefore,

$$E^2_{p,q} = \begin{cases} H_p^{cell}(X) & q=0 \\ 0 & , q \neq 0. \end{cases}$$

Now note that $\partial_r = 0$ for all $r \geq 2$, as either the domain or range is zero. Hence, $E^r_{p,q} = E^2_{p,q}$. If X is finite-dimensional, then the filtration is bounded, & so $H_p(X) = H_p^{cell}(X)$. [One can drop this assumption for example by taking a direct limit]

<u>Def^n</u>: A homological spectral sequence is a sequence

$$\{E^r_{*,*},\ d^r_{*,*}\}_{r\geq 0}$$

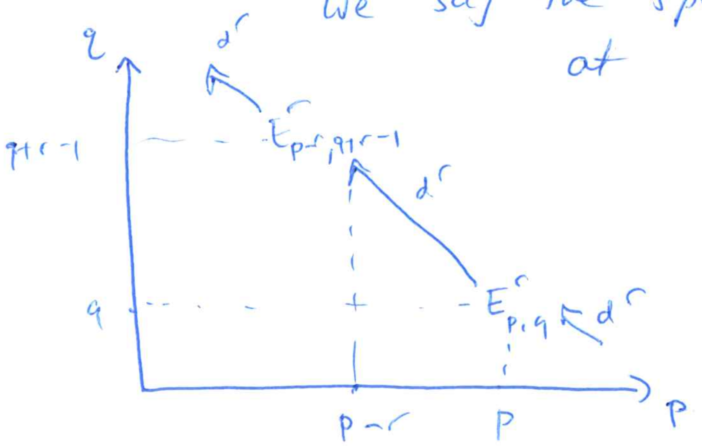
of chain complexes of abelian groups such that

$$E^{r+1}_{*,*} = H_*(E^r_{*,*}).$$

<u>Rem</u>: A spectral sequence is first quadrant if $E^r_{p,q}=0$ when $p<0$ or $q<0$. Here, for any fixed (p,q), for large enough r, the differentials vanish, & show

$$E^r_{p,q} = E^{r+1}_{p,q} = \dots E^\infty_{p,q}.$$

We say the spectral sequence degenerates at E_2.



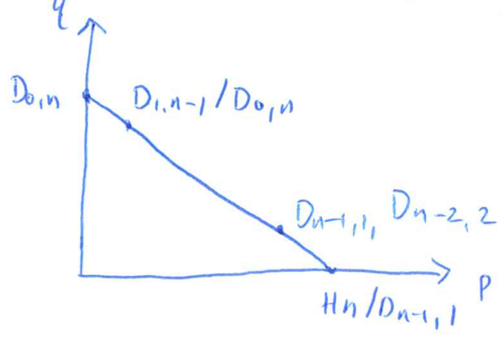
<u>Def^n</u>: If $\{H_n\}_n$ are groups we say that a spectral sequence converges/abuts to H_*, write

$$(E^r, d^r) \Rightarrow H_*$$

if for each n, there is a filtration

$$H_n = D_{n,0} \supseteq D_{n-1,1} \supseteq \dots \supseteq D_{1,n-1} \supseteq D_{0,n} \supseteq D_{-1,n+1} = 0$$

such that $$E^\infty_{p,q} = \frac{D_{p,q}}{D_{p-1,q+1}}$$



Rem:

- If $E^\infty_{p,q} = 0$ for all $p+q=n$, then $H_n = 0$
- If $H_n = 0$, then $E^\infty_{p,q} = 0$ for all $p+q=n$.

The spectral sequence of a double complex

A double complex is a bindexed family $\{C_{p,q}\}$ of abelian groups with two differentials

$$d' : C_{p,q} \to C_{p-1,q}$$
$$d'' : C_{p,q} \to C_{p,q-1}$$

such that $(d')^2 = 0$, $(d'')^2 = 0$, $d'd'' + d''d' = 0$. For simplicity, we assume $C_{p,q} = 0$ for $p<0$ or $q<0$.

To a double complex, we can associate the total complex (C_n, d), $C_n = \sum_{p+q=n} C_{p,q}$, $d = d' + d''$.

This has two filtrations

1) $'C^p_n = \sum_{\substack{j+q=n \\ j \le p}} C_{j,q}$

2) $''C^q_n = \sum_{\substack{p+k=n \\ k \le q}} C_{p,k}$.

Each filtration gives rise to a spectral sequence.

For example in (1), one has $'E^1_{p,q} = H_{p+q}('C^p/'C^{p-1})$

where $'C^p/'C^{p-1} = C_{p,n-p}$.

One checks that $'E^1$ is computed by means of d'' & d^1 is induced by d'. So we can say that

$$'E^2_{p,q} = H'_p H''_q (C) \Rightarrow H_* (C)$$

Similarly, $''E^2_{p,q} = H''_q H'_p (C) \Rightarrow H_* (C)$.