Maps of bundles

The notion of a bundle morphism is subtle, especially when there are different fibres & structure groups. One possible definition is the following.

Def^n: A morphism of fibre bundles $E \xrightarrow{\pi} B$ & $E' \xrightarrow{\pi'} B'$ with structure group G & fibre F is a pair of cts maps $f: B \to B'$, $\tilde{f}: E \to E'$ such that

(1) the diagram

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ B \arrow[r, "f"] & B' \end{tikzcd}$$

commutes.

(2) For each chart $\varphi: U \times F \to \pi^{-1}(U)$

& chart $\varphi': U' \times F \to (\pi')^{-1}(U')$

& each $b \in U$, $f(b) \in U'$

the composite
$$\{b\} \times F \xrightarrow{\varphi} \pi^{-1}(b) \xrightarrow{\tilde{f}} (\pi')^{-1}(f(b)) \xrightarrow{(\varphi')^{-1}} \{f(b)\} \times F$$
is a homeomorphism given by the action of an element $\Psi_{\varphi,\varphi'}(b) \in G$

(3) The association $b \mapsto \Psi_{\varphi,\varphi'}(b)$ defines a cts map from $U \cap f^{-1}(U')$ to G.

Rem: This is a technical definition! Note that the fibres are mapped homeomorphically by a map of fibre bundles of this type. In particular, an <u>isomorphism</u> of fiber bundles is a map of fibre bundles $(f, \tilde{f})$ which admits a map $(g, \tilde{g})$ in the reverse direction such that both compositions are the identity.

Remark: An important type of bundle map is a gauge transformation; this is a bundle map from π to itself which covers the identity of B

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, "\pi"'] & \circlearrowleft & E \arrow[dl, "\pi'"] \\ & B & \end{tikzcd}$$

By definition, such g restricts to an isomorphism given by the action of an element in the structure group on each fiber. The set of all gauge transformations forms group.

One way in which morphisms of fiber bundles arise is the following.

Def^n: Given a bundle $\pi: E \to B$ with structure group G & fiber F, & that $f: B' \to B$ is a cts map. The pull-back of π by f is the space

$$\begin{tikzcd} f^*(E) \arrow[r, "\hat{f}"] \arrow[d, "f^*\pi"'] & E \arrow[d, "\pi"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

$$f^*(E) = \{(e,x) \in E \times B' \mid \pi(e) = f(x)\}.$$

with $f^*\pi: f^*(E) \to B'$
$(e,x) \longmapsto x$

$\hat{f}: f^*(E) \to E$
$(e,x) \longmapsto e$

Theorem (a) $f^*\pi: f^*(E) \to X$ is a fiber bundle with structure group G & fiber F

(b) $(f, \hat{f})$ is a bundle map.

Proof sketch:

Let $\{\varphi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times F\}$ be an atlas of $\pi : E \to B$.

Now consider the diagram

$$\begin{tikzcd}
(f^*\pi)^{-1}(f^{-1}(\mathcal{U}_\alpha)) \arrow[r, "\hat{f}"] \arrow[d, "f^*\pi"'] & \pi^{-1}(\mathcal{U}_\alpha) \arrow[r, "\varphi_\alpha", "\cong"'] \arrow[d, "\pi"] & \mathcal{U}_\alpha \times F \arrow[dl, bend left, "pr_1"'] \\
f^{-1}(\mathcal{U}_\alpha) \arrow[r, "f"'] & \mathcal{U}_\alpha &
\end{tikzcd}$$

What are the elements of $(f^*\pi)^{-1}(f^{-1}(\mathcal{U}_\alpha))$?

$$= \{(x,e) \in f^{-1}(\mathcal{U}_\alpha) \times \underbrace{\pi^{-1}(\mathcal{U}_\alpha)}_{\cong \mathcal{U}_\alpha \times F} \mid f(x) = \pi(e)\}$$

Define $k_\alpha : (f^*\pi)^{-1}(f^{-1}(\mathcal{U}_\alpha)) \longrightarrow f^{-1}(\mathcal{U}_\alpha) \times F$

by $\qquad (x,e) \longmapsto (x, pr_2 \circ \varphi_\alpha(e))$

We can also define

$$k_\alpha^{-1} : f^{-1}(\mathcal{U}_\alpha) \times F \longrightarrow (f^*\pi)^{-1}(f^{-1}(\mathcal{U}_\alpha))$$
$$(x, m) \longmapsto (x, \varphi_\alpha^{-1}(f(x), m))$$

These are inverse homeomorphisms, e.g.,

$$\begin{aligned} k_\alpha^{-1} \circ k_\alpha &= k_\alpha^{-1}(x, pr_2 \circ \varphi_\alpha(e)) \\ &= (x, \varphi_\alpha^{-1}(f(x), pr_2 \circ \varphi_\alpha(e))) \\ &= (x, \varphi_\alpha^{-1}(\pi(e), pr_2 \circ \varphi_\alpha(e))) \end{aligned}$$

So it suffices to show $\varphi_\alpha^{-1}(\pi(e), pr_2 \circ \varphi_\alpha(e)) = e$

$$= \{e' \in \pi^{-1}(\mathcal{U}_\alpha) \mid \varphi_\alpha(e') = (\pi(e), pr_2 \circ \varphi_\alpha(e))\}$$
$$= e$$

$\Rightarrow \{(f^{-1}(\mathcal{U}_\alpha), k_\alpha)\}_\alpha$ is a trivializing atlas for $f^*\pi$.

– We now show that the structure group is G. To ease notation, let $V_\alpha = f^{-1}(\mathcal{U}_\alpha)$. We want to show that the transition functions for $f^*(E) \to B'$ are given by $\{\Phi^{\alpha,\beta} \circ f\}$.

For $(x, m) \in (U_\alpha \cap V_\beta) \times F$, we have

$$k_\beta \circ k_\alpha^{-1}(x, m) = k_\beta \circ \left(x, \varphi_\alpha^{-1}(f(x), m)\right)$$
$$= \left(x, pr_2 \circ \varphi_\beta \circ \varphi_\alpha^{-1}(f(x), m)\right)$$

Let $\Phi^{\alpha,\beta} : U_{\alpha,\beta} \longrightarrow \mathrm{Homeo}(F)$ be a coordinate transform of $\pi : E \to B$, so that

$$pr_2 \circ \varphi_\beta \circ \varphi_\alpha^{-1}(f(x), m) = \Phi^{\alpha,\beta}(f(x))(m)$$

$$\Rightarrow k_\beta \circ k_\alpha^{-1}(x, m) = \left(x, \Phi^{\alpha,\beta}(f(x))(m)\right) \qquad \square$$

This shows (a), & we omit (b).

We have the following important result, whose proof we return to later.

<u>Theorem</u>: Given a fiber bundle $\pi : E \to B$ with structure group G & fiber F, & two homotopic maps $f, g : B' \to B$, then $f^*\pi \cong g^*\pi$ as bundles over B.

<u>Rem</u>: One checks that f^* is functorial.

<u>Cor (Exercise)</u>: A fiber bundle over a contractible base space is trivial.

<u>Exercise</u>: Let

$$\begin{tikzcd} E' \arrow[r, "\hat{f}"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

be a bundle map, then there is a factorization

$$\begin{tikzcd} E' \arrow[r, "\beta'"] \arrow[dr, "\pi'"'] & f^*E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ & B' \arrow[r] & B \end{tikzcd}$$

so that $f^* \circ \beta = \hat{f}$ with (β, Id) a map of bundles over B'.

~~B~~ Classification of bundles

Example: We want to classify iso. classes of bundles ~~of~~ over S^n with structure group G, & fiber F.

Claim: These are isomorphic to $\pi_{n-1}(G)$.

Indeed, cover S^n with two contractible sets U_+, U_- obtained by removing the north & south poles. Let $i_\pm : \mathcal{U}_\pm \hookrightarrow S^n$ be the inclusion. Any bundle π over S^n is contractible when restricted to $\mathcal{U}_\pm$, ie $i_\pm^* \pi \cong \mathcal{U}_\pm \times F$. Then, the bundle is completely determined by a map

$$g_\pm : \mathcal{U}_+ \cap \mathcal{U}_- \simeq S^{n-1} \longrightarrow G, \quad \text{ie}$$

by an element of $\pi_{n-1}(G)$.

Let $\mathcal{B}(X, G, F, e)$ denote iso classes (over id_X) of fiber bundles on X with structure group G, fiber F, & G-action given by e. We let

$$\mathcal{P}(X, G) := \mathcal{B}(X, G, G, \mu)$$

be the principal G-bundles, where $\mu : G \times G \to G$ is the group multiplication.

The associated bundle construction

Given a G-space F, we construct a map

$$\mathcal{P}(X, G) \longrightarrow \mathcal{B}(X, G, F, e)$$

as follows: start with a principal G-bundle $\pi : E \to X$. By Friday, G acts freely on the right of E. Since G acts on the left on F, we can define a left action on $E \times F$ by

$$g \cdot (e, f) = (e \cdot g^{-1}, g \cdot f).$$

Let $E \times_G F := E \times F / G$ be the quotient space.

there is a projection map $\omega : E\times_G F \longrightarrow E/G \cong X$.

Def^n: The projection $\omega := \pi\times_G F : E\times_G F \to X$ is called the associated bundle with fiber F.

Before we prove this is actually a bundle, let us do an example.

Example: Let $\pi: S^1 \longrightarrow S^1$, $z \longmapsto z^2$. By covering space theory, this is a principal $\mathbb{Z}/2$-bundle. ~~Each~~ Let $F = [-1,1]$, & let $\mathbb{Z}/2 = \{-1,1\}$ act on F by multiplication.

Then the bundle associated to π with fiber F is the Möbius strip $S^1 \times_{\mathbb{Z}/2} [-1,1] \cong \dfrac{S^1\times[-1,1]}{(x,t)\sim(a(x),-t)}$

for $a: S^1 \to S^1$ the antipodal map. Similarly, the bundle associated to π with fiber F is the Klein bottle.

Thm $\omega : E\times_G F \longrightarrow X$ is a fiber bundle with group G, fiber F, & having the same transition functions as π.

Moreover, the assignment

$$\pi \longmapsto \omega$$

defines a bijection $P(X,G) \to B(X,G,F,e)$.

Sketch proof: Let $\{\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G\}$ be local trivializations of a principal G-bundle $\pi: E \to X$. We want to use these to construct local trivializations $\{\psi_\alpha : \omega^{-1}(U_\alpha) \to U_\alpha \times F\}$ of ω. We have

$$\begin{aligned}\omega^{-1}(U_\alpha) &= \{[e,f] \in E\times_G F \mid \omega([e,f]) \in U_\alpha\}\\ &= \{[e,f] \in E\times_G F \mid \pi(e) \in U_\alpha\}\\ &= \pi^{-1}(U_\alpha)\times_G F\end{aligned}$$

So we get maps

$$\omega^{-1}(U_\alpha) = \pi^{-1}(U_\alpha) \times_G F \xrightarrow{\varphi_\alpha \times_G F} (U_\alpha \times G) \times_G F.$$

But
$$(U_\alpha \times G) \times_G F \longrightarrow U_\alpha \times F$$
$$[(x,g), f] \longmapsto (x, gf)$$
is a homeomorphism, with inverse $(x,y) \longmapsto ([x,e], f)$.

We define the required map $\psi_\alpha : \omega^{-1}(U_\alpha) \to U_\alpha \times F$ as the composite

$$\omega^{-1}(U_\alpha) = \pi^{-1}(U_\alpha) \times_G F \xrightarrow[\cong]{\varphi_\alpha \times_G \mathrm{id}_F} (U_\alpha \times G) \times_G F \xrightarrow[1 \times \mu]{\cong} U_\alpha \times F$$

From the definition, the diagram

$$\begin{tikzcd} \omega^{-1}(U_\alpha) \arrow[rr, "\psi_\alpha"] \arrow[dr, "\omega"'] & & U_\alpha \times F \arrow[dl, "pr_1"] \\ & U_\alpha & \end{tikzcd}$$

commutes.

In order to show it has structure group G, let $(x,y) \in (U_\alpha \cap U_\beta) \times F$

$$\begin{aligned}
\psi_\beta \circ \psi_\alpha^{-1}(x,y) &= \psi_\beta \circ (\varphi_\alpha^{-1} \times_G 1_F) \circ (1 \times \mu)^{-1}(x,y) \\
&= \psi_\beta \circ (\varphi_\alpha^{-1} \times_G 1_F)(x, e, y) \\
&= \psi_\beta(\varphi_\alpha^{-1}(x,e), y) \\
&= (1 \times \mu) \circ (\varphi_\beta \times_G 1_F)(\varphi_\alpha^{-1}(x,e), y) \\
&= (1 \times \mu)(\varphi_\beta \circ \varphi_\alpha^{-1}(x,e), y) \\
&= (1 \times \mu)(x, \Phi^{\alpha,\beta}(x)e, y) \\
&= (x, \Phi^{\alpha,\beta}(x)y)
\end{aligned}$$

where $\Phi^{\alpha,\beta} : U_\alpha \cap U_\beta \to G$ is a coordinate transform of π. $\square$.

Thm: Let $\pi : E \to Y$ be a fiber bundle with structure group G & fiber F & let $f \simeq g : X \to Y$ be two homotopic maps. Then $f^*\pi \cong g^*\pi$ over id_X.

Proof: We can assume our bundles are principal

$$\begin{tikzcd} f^*E \arrow[rr, dashed, "?"] \arrow[dr] & & g^*E \arrow[dl] \\ & X & \end{tikzcd}$$

So we want to understand bundle maps b/w principal G-bundles. So, say we have the following:

$$\begin{tikzcd} E_1 \arrow[r, "\tilde{f}"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

G acts on the right of E_1 & E_2, $\Rightarrow$ G acts via the left as well, ie. E_2 is a left G-space. We can take the associated bundle of π_1 with fiber E_2, $\omega : E_1 \times_G E_2 \longrightarrow X$

Def^n: Given a bundle $\pi : E \to B$, a section s is a cts map $s : B \to E$ such that $\pi \circ s \simeq id_B$.

Theorem: Bundle maps from π_1 to π_2 are in one-to-one correspondence with sections of ω.

Proof: We can prove this locally, & so it suffices to assume the bundles are trivial. So let $U \subseteq Y$ be open, & $V \subseteq f^{-1}(U)$ be open, so that the following diagram commutes

$$\begin{tikzcd} V \times G \arrow[r, "\tilde{f}"] \arrow[d, "\pi_1"'] & U \times G \arrow[d, "\pi_2"] \\ V \arrow[r, "f"'] & U \end{tikzcd}$$

We define a section in

$$\begin{tikzcd} (V\times G)\times_G (U\times G) \arrow[d, shift left, "\omega"] \\ V \arrow[u, shift left, bend left, "\sigma"] \end{tikzcd}$$

as follows: for $e_1 \in V\times G$, with $x = \pi_1(e_1)\in V$, we set

$$\sigma(x) = [e_1, \hat{f}(e_1)]$$

This is well-defined, since for $g\in G$, we have

$$\begin{aligned} [e_1\cdot g, \hat{f}(e_1\cdot g)] &= [e_1\cdot g, \hat{f}(e_1)\cdot g] \\ &= [e_1\cdot g, g^{-1}\cdot \hat{f}(e_1)] \\ &= [e_1, \hat{f}(e_1)] \end{aligned}$$

σ is cts, &

$$\begin{aligned} \omega\sigma(x) &= \omega[e_1, \hat{f}(e_1)] \\ &= \pi_1(e_1) \\ &= x \end{aligned}$$

Conversely, given a section of $E_1\times_G E_2 \xrightarrow{\omega} X$, define a bundle map $(f,\hat{f})$ by

$$\hat{f}(e_1) = e_2$$

where $\sigma(\pi_1(e_1)) = [(e_1, e_2)]$.

This map is equivariant

$$\begin{aligned} [e_1\cdot g, e_2\cdot g] &= [e_1\cdot g, g^{-1}\cdot e_2] \\ &= [e_1, e_2] \end{aligned}$$

have $\hat{f}(e_1\cdot g) = e_2\cdot g = \hat{f}(e_1)\cdot g$, & so $\hat{f}$ descends to a map on orbit spaces, which gives a bundle map.

Lem: Let $\pi : E \to X\times I$ be a principal bundle, & let $\pi_0 := i_0^*\pi$, $E_0\to X$, be the pullback of π under $i_0 : X \to X\times I$, $x\mapsto (x,0)$.
Then, $\pi \cong (pr_1)^*\pi_0 \cong \pi_0\times id_I$, where $pr_1 : X\times I\to X$ is the projection map.

Proof:

$$\begin{tikzcd}
E_0 \arrow[r, "\tilde{i}_0"] \arrow[d, "\pi_0"'] & E \arrow[r, dashed, "\widetilde{pr}_1"] \arrow[d] & E_0 \arrow[d] \\
X \arrow[r, hook, "i_0"'] \arrow[rr, bend right, dashed, "id_X"'] & X \times I \arrow[r, "pr_1"'] & X
\end{tikzcd}$$

We want to find a bundle map $(\widetilde{pr_1}, pr_1)$.

This is equivalent to finding a section of

$$\omega : E \times_q E_0 \longrightarrow X \times I$$

Now $\omega_0 : E_0 \times_q E_0 \longrightarrow X = X \times \{0\}$ has a section σ_0 corresponding to the bundle map $(id_X, id_{E_0}) : \pi_0 \to \pi_0$.

$[\omega_0 \circ \sigma_0 \simeq id_X]$

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\sigma"] \arrow[d, hook] & E \times_q E_0 \arrow[d, "\omega"] \\
X \times I \arrow[r, "id"'] \arrow[ur, dashed, "s"] & X \times I
\end{tikzcd}$$

so that $\omega \circ s \simeq id_{X \times I}$.

Proof of the theorem: Let $H : X \times I \to Y$ be a homotopy between f & g with $H(x,0) = f(x)$ & $H(x,1) = g(x)$. Let $H^*\pi$ be the induced bundle over $X \times I$. Then we have the following diagram:

$$\begin{tikzcd}
f^*E \arrow[rr] \arrow[d, "f^*\pi"'] & & H^*E \arrow[r, "\tilde{H}"] \arrow[d, "H^*\pi"] & E \arrow[d, "\pi"] \\
& g^*E \arrow[ur] \arrow[d, "g^*\pi"] & & \\
X \times \{0\} \arrow[rr, "i_0"] & X \times \{1\} \arrow[r, "i_1"] & X \times I \arrow[r] \arrow[dr, "pr_1"] & Y \\
& & & X
\end{tikzcd}$$

Since $f = H(-,0)$, we get $f^*\pi = i_0^* H^*\pi$. By the lemma, $H^*\pi \simeq pr_1^*(f^*\pi) \simeq pr_1^*(g^*\pi)$, & thus

$$f^*\pi \simeq i_0^* H^*\pi \simeq i_0^* pr_1^* g^*\pi \simeq g^*\pi.$$