Pontryagin classes

Recall the following from last week:
If ω is a complex vector bundle, then

$$c_k(\overline{\omega}) = (-1)^k c_k(\omega)$$

If π is a real vector bundle

$$c_k(\pi \otimes \mathbb{C}) = c_k(\overline{\pi \otimes \mathbb{C}}) = (-1)^k c_k(\pi \otimes \mathbb{C})$$

so $c_k(\pi \otimes \mathbb{C})$ is of order 2 ~~2-torsion~~ if k is odd.

<u>Def^n</u>: Let $\pi : E \to X$ be a real vector bundle of rank n. The i-th Pontryagin class of π is defined as

$$p_i(\pi) := (-1)^i c_{2i}(\pi \otimes \mathbb{C}) \in H^{4i}(X;\mathbb{Z})$$

If ω is a complex vector bundle

$$p_i(\omega) := p_i(\omega_{\mathbb{R}}) = (-1)^i c_{2i}(\omega \oplus \overline{\omega})$$

<u>Remark</u>: $p_i(\pi) = 0$ for all $i > n/2$.

<u>Defn</u>: The total Pontryagin class is

$$p(\pi) = 1 + p_1(\pi) + \dots \in H^*(X;\mathbb{Z})$$

We would like Pontryagin classes to satisfy a product formula. Since we have ignored the odd classes, the best we get is the following:

<u>Theorem</u>: $p(\pi_1 \oplus \pi_2) = p(\pi_1) \smile p(\pi_2)$ modulo 2-torsion.

In other words, $2(p(\pi_1 \oplus \pi_2) - p(\pi_1) \smile p(\pi_2)) = 0$.

<u>Proof</u>: $(\pi_1 \oplus \pi_2) \otimes \mathbb{C} = (\pi_1 \otimes \mathbb{C}) \oplus (\pi_2 \otimes \mathbb{C})$

So,

$$p_i(\pi_1 \oplus \pi_2) = (-1)^i c_{2i}((\pi_1 \oplus \pi_2) \otimes \mathbb{C})$$
$$= (-1)^{2i} c_{2i}((\pi_1 \otimes \mathbb{C}) \oplus (\pi_2 \otimes \mathbb{C}))$$

Now $c_{2i}[(\pi_1 \otimes \mathbb{C}) \oplus (\pi_2 \otimes \mathbb{C})] = \sum_{k+l=2i} c_k(\pi_1 \otimes \mathbb{C}) \smile c_l(\pi_2 \otimes \mathbb{C})$

$$c_{2i}((\pi_1 \oplus \pi_2)\otimes \mathbb{C}) = \sum_{k+l=i} c_{2k}(\pi_1 \otimes \mathbb{C}) \smile c_{2l}(\pi_2 \otimes \mathbb{C}) \quad \text{mod 2-torsion}$$

$$\Rightarrow p_i(\pi_1 \oplus \pi_2) = \sum_{k+l=i} p_k(\pi_1) \smile p_l(\pi_2)$$

<u>Def</u>: If M is a real smooth manifold, then

$$p(M) := p(TM)$$

If M is a complex manifold

$$p(M) = p((TM)_{\mathbb{R}}).$$

Note that the ~~Pontryagin~~ chern classes of a complex bundle determine ~~those of~~ the Pontryagin classes.

<u>Example</u>: Recall $c(\mathbb{C}P^n) = (1+c)^{n+1}$ for $c \in H^2(\mathbb{C}P^n;\mathbb{Z})$.

Then $c(\overline{T\mathbb{C}P^n}) = (1-c)^{n+1}$

$$\begin{aligned} c((T\mathbb{C}P^n)_{\mathbb{R}} \otimes \mathbb{C}) &= c(T\mathbb{C}P^n \oplus \overline{T\mathbb{C}P^n}) \\ &= (1+c)^{n+1}(1-c)^{n+1} \\ &= (1-c^2)^{n+1} \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k}(-c^2)^k = \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} c^{2k} \end{aligned}$$

$$p_k(T\mathbb{C}P^n) = \binom{n+1}{k} c^{2k}$$

ie $p(T\mathbb{C}P^n) = (1+c^2)^{n+1}$

e.g.

$$\begin{aligned} p(T\mathbb{C}P^1) &= 1 \\ p(T\mathbb{C}P^2) &= 1 + 3c^2 \\ p(T\mathbb{C}P^3) &= 1 + 4c^2 \\ p(T\mathbb{C}P^4) &= 1 + 5c^2 + 10c^4 \end{aligned}$$

Application to the embedding problem

After forgetting the complex structure $\mathbb{C}P^n$ is a $2n$-dimensional real smooth manifold. Suppose we have an embedding

$$\mathbb{C}P^n \hookrightarrow \mathbb{R}^{2n+k}$$

Then, as on Tuesday, there exists a normal bundle ν^k of rank k such that

$$T\mathbb{C}P^n \oplus \nu^k \cong \varepsilon^{2n+k}$$

Using the product formula & noticing $H^*(\mathbb{C}P^n; \mathbb{Z})$ has no 2-torsion we get

$$p(\mathbb{C}P^n) \cdot p(\nu^k) = 1 .$$

Since ν has rank k, we have $p_i(\nu^k) = 0$ for $i > k/2$

i.e. if $p_i(\nu^k) \neq 0$, then $i \leq k/2$.

Example: $p(\mathbb{C}P^2) = 1 + 3c^2$

Suppose we have an embedding $\mathbb{C}P^2 \hookrightarrow \mathbb{R}^{4+k}$

$$p(\nu^k)(1+3c^2) = 1$$

$$\Rightarrow p(\nu^k) = 1 - 3c^2 \qquad \text{(because } c^4 = 0\text{)}.$$

i.e. $p_1(\nu^k) \neq 0 \Rightarrow 1 \leq k/2$

i.e., $k \geq 2$.

so $\mathbb{C}P^2$ does not embed into $\mathbb{R}^5$.