Let $\mathcal{E}$ be a category with pushouts & binary products. When does taking product preserve pushouts?

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} \qquad E \in \mathcal{E} \quad \text{is} \qquad \begin{tikzcd} A \times E \arrow[r] \arrow[d] \arrow[dr, phantom, "(*)"] & B \times E \arrow[d] \\ C \times E \arrow[r] & D \times E \end{tikzcd}$$

If $- \times E$ has a right adjoint, then $(*)$ is still a pushout. A pushout is a colimit, & left adjoints preserve ~~the~~ colimits.

e.g. $\mathcal{E}$ = "nice topological spaces" [CWGH "compactly generated weak Hausdorff"]

$F : \mathcal{E} \longrightarrow D$

$G : D \longrightarrow \mathcal{E}$

(F, G) are adjoint $\mathrm{Hom}_D(F(M), N) \cong \mathrm{Hom}_{\mathcal{E}}(M, G(N))$ (actual iso)

for all $M \in \mathcal{E}$, $N \in D$

Then F is left adjoint to G. G is right adjoint to F.

$$\mathrm{Hom}_{Sets}(A \times B, C) = \mathrm{Hom}_{Sets}(A, C^B)$$

Homotopy extension & lifting property

Recap of what HELP property. Given a diagram:

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d] & X \arrow[d, "f"] \arrow[dl, dashed, "\tilde{g}"'] \\
Y \arrow[r, hook, "e"'] & Z
\end{tikzcd}$$

where $e : Y \longrightarrow Z$ is a n-equivalence, (X, A) is a relative CW-complex ~~of~~ of $\dim \leq n$ such that this diagram commutes up to a homotopy H, then there exists

$\tilde{g} : X \longrightarrow Y$

such that

- the top triangle commutes strictly

- the low triangle commutes up to a homotopy $\tilde{H}$ extending H $\tilde{H}|_{A\times I} \simeq H$.

<u>How do we get HLP from this?</u>

Good question!

<u>Thm</u>: If X is a CW-complex, & $e: Y \longrightarrow Z$ is an n-equivalence, then

$$e_*: [X, Y] \longrightarrow [X, Z]$$

~~this~~ is a bijection if $\dim(X) < n$, & a surjection if $\dim(X) = n$.

<u>Proof</u>: Let $[f] \in [X, Z]$, apply HELP to $(X, \emptyset)$:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X \arrow[d, "f"] \arrow[dl, dashed, "\tilde{g}"'] \\
Y \arrow[r, "e"'] & Z
\end{tikzcd}$$

By HELP, there exist $\tilde{g}: X \longrightarrow Y$ such that $\underset{\simeq f}{e \circ \tilde{g}} : X \longrightarrow Z$

i.e. $e_*[\tilde{g}] = [f]$

i.e., e_* is surjective.

Injectivity: Assume $[g_0], [g_1] \in [X, Y]$ with

$$e[g_0] = e_*[g_1]$$

i.e., $e g_0 \simeq e g_1$ via some homotopy

$$F : X \times I \longrightarrow Z$$

Consider the pair $(X \times I, X \times \partial I)$ and the map $g : X \times \partial I \longrightarrow Y$

$$(x, \nu) \longmapsto g_\nu(x) \qquad \nu = 0, 1$$

$$\begin{tikzcd}
X \times \partial I \arrow[r] \arrow[d, "g"'] & X \times I \arrow[d, "F"] \\
Y \arrow[r, "e"'] & Z
\end{tikzcd}$$

Define $H : (X \times \partial I) \times I \longrightarrow Z$

$$(x, i, s) = e \circ g_i(s) \qquad i = 0, 1, \; s \in I$$

Applying HELP we get $\tilde{g} : X \times I \longrightarrow Y$ which is a homotopy between g_0 and g_1.

$\Rightarrow [g_0] = [g_1]$

$\Rightarrow e_*$ is injective.

Cor: If X is a CW-complex, $e: Y \longrightarrow Z$ is a weak homotopy equivalence, then
$$e_*: [X, Y] \longrightarrow [X, Z] \text{ is a bijection.}$$

Thm (Whitehead's theorem) A weak homotopy equivalence b/w CW-complexes is a homotopy equivalence.

✯ **Rem:** A weak homotopy equivalence requires a map! We have seen examples of spaces with isomorphic homotopy groups that are not homotopy equivalent. ✯

Proof: Let $e: Y \longrightarrow Z$ be a weak equivalence. Then $e_*: [Z, Y] \longrightarrow [Z, Z]$ is a bijection, so there exists $f: Z \longrightarrow Y$ such that $e \circ f \simeq id_Z$. Then $e \circ f \circ e \simeq e$, but we also have

$$(*) \quad e_*: [Y, Y] \xrightarrow{\simeq} [Y, Z]$$

this implies

$\Rightarrow f \circ e \simeq id_Y$, as $f \circ e$ & id_Y are mapped to the same element under the bijection $(*)$ $\square$.

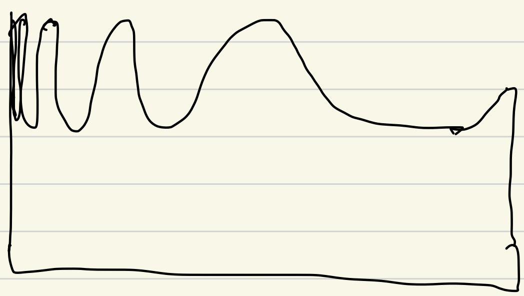
Remark: (The polish circle / the topologists sine curve)

$$A = \{ (x,y) \in \mathbb{R}^2 \mid 0 < x \leq 1, \ y = \sin(\pi/x) \}$$

$$B = \{ (0,y) \in \mathbb{R}^2 \mid -3/2 \leq y \leq 1 \}$$

$$C = \{ (0,y) \mid y \in [-3/2, -1] \} \cup \{ (x, -3/2) \in \mathbb{R}^2 \mid x \in [0,1] \} \cup \{ (1,y) \mid y \in [-3/2, 0] \}$$

$$X = A \cup B \cup C$$



Then $\pi_n X = 0$ for all n but X is not contractible.

Excision

Def^n: A triad $(X; A, B)$ is a space X, together with subspaces $A, B \subset X$. It is excisive if $X = \underbrace{A^\circ}_{\text{interior of } A} \cup B^\circ$

Rem: Do not confuse a triad $(X; A, B)$ with a triple (X, A, B).

Thm: Let $(X; A, B)$ be an excisive triad such that $C = A \cap B$ is non-empty. Assume that (A, C) is $(m-1)$-connected & (B, C) is $(n-1)$-connected where $m \geq 2$, $n \geq 1$.

Then, the inclusion $(A, C) \longrightarrow (X, B)$ is an $(m+n-2)$-equivalence

[$\pi_q(A, C) \longrightarrow \pi_q(X, B)$ is an iso for $q < m+n-2$ & surjection for $q = m+n-2$.].

Sketch of proof:

① Homotopy long exact sequence of a triple

For a triple (Y, S, T) & any basepoint there is a long exact sequence

$$\cdots \longrightarrow \pi_q(S, T) \xrightarrow{i_*} \pi_q(Y, T) \xrightarrow{j_*} \pi_q(Y, S) \longrightarrow \pi_{q-1}(S, T) \longrightarrow \cdots$$

② Define triad homotopy groups with basepoint $* \in C = A \cap B$.

$\underline{\text{Def}^n}$: $\pi_q(X; A, B) := \pi_{q-1}(P(X, *, B), P(A, *, C))$

where $P(X, *, B)$ = paths in X starting at $*$ and ending in B

$= \{ \gamma \in PX \mid \gamma(1) \in B \}$

whee $PX = \{ \gamma \in X^I \mid \gamma(0) = x_0 \}$

(3) Combine (1) and (2) to get a long exact sequence

$$\cdots \longrightarrow \pi_{q+1}(X; A, B) \longrightarrow \pi_q(A, C) \longrightarrow \pi_q(X, B)$$
$$\longrightarrow \pi_q(X; A, B) \longrightarrow \cdots$$

& then homotopy excision is the claim that

$\pi_q(X; A, B) = 0$ for $2 \leq q \leq m+n-2$.

(4) Reduction to the case where (X, A, B) is a $\underline{\text{CW-triad}}$ (A, B are subcomplexes).

(5) Finally, reduce to $A = C \cup e^m$
$B = C \cup e^n$

& then prove directly in this case.

["A concise course in algebraic topology"
Chapter 11

Geometric proof in Hatcher's book.