Week 2

Recap on CW-complexes.

Recall that a CW-complex is built via the following inductive process:

(1) Start with a discrete set X^0, the zeo cells of X.

(2) Inductively form X_n from X_{n-1} by attaching n-cells e_α^n via maps

$$\varphi_\alpha : S^{n-1} \longrightarrow X_{n-1}$$

$$
\begin{tikzcd}
\coprod S^{n-1} \arrow[r, "(\varphi_\alpha)"] \arrow[d, "i"'] & X_{n-1} \arrow[d] \\
\coprod D^n \arrow[r] & X_n = X_{n-1} \cup_{(\varphi_\alpha)} (\coprod D^n) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

[Recall: a pushout is

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \arrow[ddr, bend left] & \\
C \arrow[r] \arrow[drr, bend right] & Z \arrow[dr, dashed, "\exists!"] \arrow[ul, phantom, "\ulcorner", very near start] & \\
& & X
\end{tikzcd}
$$
]

(3) $X = \bigcup_n X^n$ with the weak topology

($A \subset X$ is open/closed $\Leftrightarrow$ $A \cap X^n$ is open/closed for each n).

Rem: A relative CW-complex (X, A) has $X_0 = \{\text{discrete set of points}\} \cup A$.

Def^n: A map of pairs $f:(X,A) \to (Y,B)$ b/w relative CW complexes is cellular if $f(X_n) \subset Y_n$ for all n.

Theorem (cellular approximation theorem)
Any map $f:(X,A) \longrightarrow (Y,B)$ between relative CW-complexes is homotopic rel A to a cellular map.

Exercise: Use cellular approximation to compute $\pi_i(S^n)$ for $i<n$.

Cofibrations & HEP

Def^n (Homotopy Extension Property) Let $\mathcal{C}$ be a class of topological spaces. A map $i: A \longrightarrow X$ has the $\mathcal{C}$-HEP if, for every $Y \in \mathcal{C}$, the following extension problem has a solution

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i \times id"] \arrow[ddr, bend left, "H"] & \\
X \arrow[r, "i_0"'] \arrow[drr, bend right, "f"'] & X \times I \arrow[dr, dashed, "\exists \tilde{H}"] & \\
& & Y
\end{tikzcd}$$

$\tilde{H}\big|_{A \times I} = H$

$\boxed{Hom(A \times B, C) = Hom(A, Map(B, C))}$

Rem: $\tilde{H}$ need not be unique

Rem: We can rewrite this diagram as the following

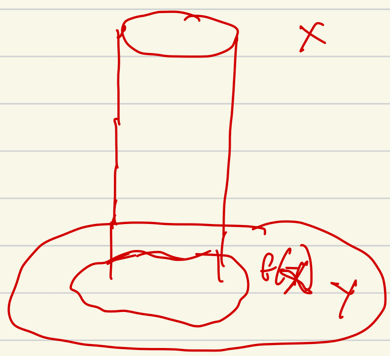
$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "i"'] & Y^I = Map(I, Y) \arrow[d, "ev_0"] \\
X \arrow[r, "f"'] \arrow[ur, dashed, "\exists \tilde{h}"] & Y
\end{tikzcd}$$

Def^n: If $\mathcal{C}$ is the class of all topological spaces, if $i : A \longrightarrow X$ has the $\mathcal{C}$-HEP, then i is called a cofibration.

Examples $J = [0, 1]$

The mapping cylinder associated to a map $X \xrightarrow{f} Y$ is the pushout

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[d, "f"'] & X \times J \arrow[d] \\ Y \arrow[r] & M_f \end{tikzcd}$$



M_f deformation retracts onto Y by sliding each point $(x,t) \in M_f$ ~~at~~ to the endpoint $f(x)$.

(i) $i_0 : X \longrightarrow X \times J$ is a cofibration

(ii) $i_0 : X \longrightarrow CX$ is a cofibration

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \times I \arrow[d] \arrow[rd, bend left, "H"] & \\ X \times J \arrow[r] \arrow[rr, bend right, "f"'] & (X \times J) \times I \arrow[r, dashed, "\tilde{H}"] & Y \end{tikzcd}$$

$$\tilde{H}((x,t),s) = \begin{cases} f(x, 1-(1-t)(1+s)) & (1-t)(1+s) \le 1 \\ H(x, (1-t)(1+s)-1) & (1-t)(1+s) \geq 1 \end{cases}$$

"stretch the cylinder as the cone through the homotopy"

<u>$t=0$</u> $(1-t)(1+s) = (1+s) \geq 1$, so

$$\tilde{H}((x,0), s) = H(x,s)$$

<u>$t=1/4$</u> $(1-t)(1+s) = \frac{3}{4}(1+s) \le 1 \Leftrightarrow s \le 1/3$

$$\tilde{H}((x,\tfrac{1}{4}), s) = \begin{cases} f(x, \frac{1}{4}(1-3s)) & s \le \frac{1}{3} \\ H(x, \frac{1}{4}(3s-1)) & s \geq \frac{1}{3} \end{cases}$$

$$f(x,\tfrac{1}{4}) \rightsquigarrow H(x, \tfrac{1}{2})$$

<u>$t=\frac{1}{2}$</u> $(1-t)(1+s) = \frac{1}{2}(1+s) \le 1$ for all s

$$\tilde{H}((x,\tfrac{1}{2}), s) = f(x, \tfrac{1}{2}(1-s))$$

$$f(x,\tfrac{1}{2}) \rightsquigarrow f(x,0)$$

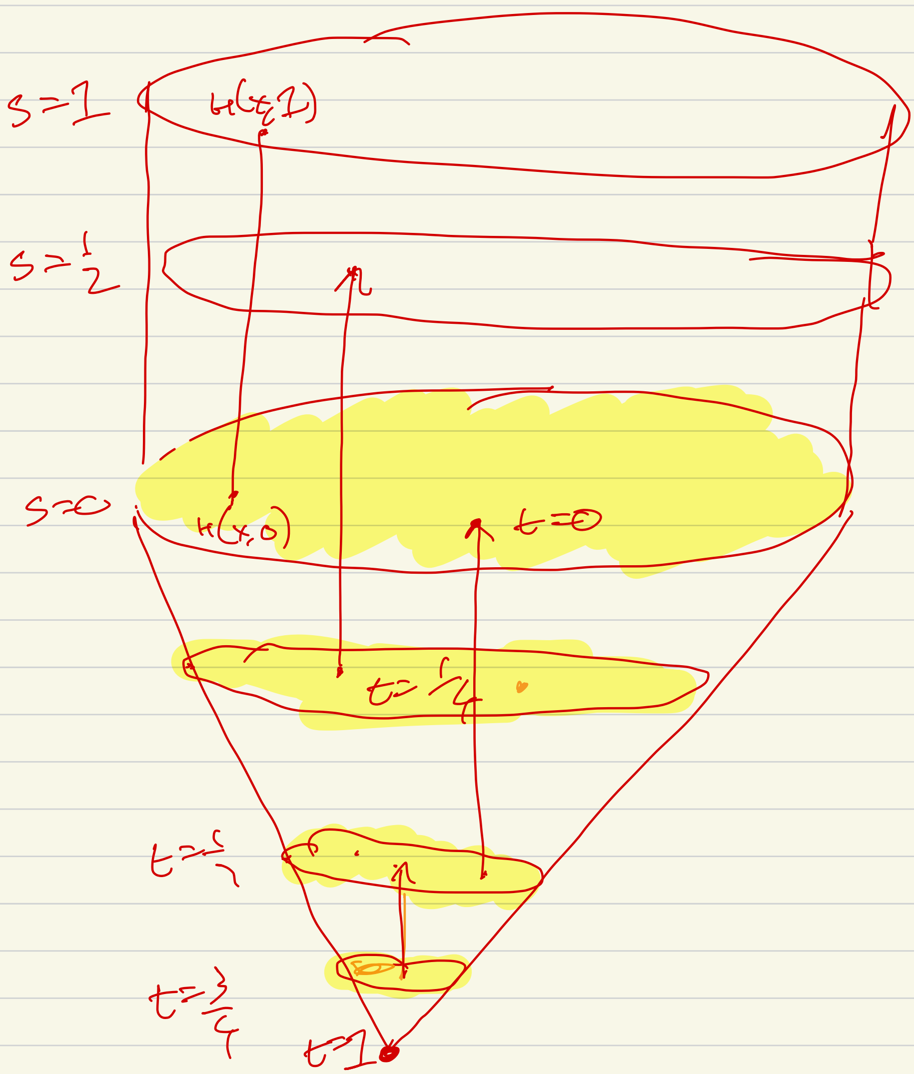
$\underline{t = \frac{3}{4}}$ $\quad (1-t)(1+s) = \frac{1}{4}(1+s) \leq 1$ for all s

$$\tilde{H}((x, \tfrac{1}{2}), s) = f(x, \tfrac{1}{4}(3-s))$$

$$f(x, \tfrac{3}{4}) \rightsquigarrow f(x, \tfrac{1}{2})$$

$\underline{t=1}$ $\quad (1-t)(1+s) = 0 \leq 1$

$$\tilde{H}((x, 1), s) = f(x, 1).$$



$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & M_f \arrow[ur, "\simeq"'] & \end{tikzcd}$$

(iii) $X \longrightarrow M_f$ is a cofibration.

Cor: $S^{n-1} \longrightarrow D^n$ is a cofibration.

Prop: (Universal test diagram) Let $i: A \longrightarrow X$, let M_i be the mapping cylinder of i.

Then $i: A \longrightarrow X$ is a cofibration $\Longleftrightarrow \exists\, r: X \times I \longrightarrow M_i$ making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i \times id"] \arrow[ddr, bend left, "J"] & \\ X \arrow[r, "i_0"] \arrow[drr, bend right, "j"'] & X \times I \arrow[dr, dotted, "r"] & \\ & & M_i \end{tikzcd}$$

Proof: ($\Rightarrow$) If $i: A \longrightarrow X$ is a cofibration, then this is just HEP.

(⇐)
Conversely, suppose such a map exists.

Consider the following diagram:

$$
\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "i"'] & A\times I \arrow[d, "i\times id"] \arrow[ddr, bend left] \arrow[dddrr, bend left, "H"] & & \\
X \arrow[r, "i_0"'] \arrow[drr, bend right] \arrow[ddrrr, bend right, "f"'] & X\times I \arrow[dr, "r"] & & \\
 & & M_i \arrow[dr, dashed, "H'"] & \\
 & & & Y
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & A\times I \arrow[d] \arrow[ddr, bend left, "H"] & \\
X \arrow[r] \arrow[drr, bend right, "f"'] & M_i \arrow[dr, dashed] & \\
 & & Y
\end{tikzcd}
$$

Define $\tilde{H} = H' \circ r$

Cor: If $A \subset X$, then $i: A \to X$ is a cofibration $\Leftrightarrow$ $X\times I$ is a retract of M_i.

Cor: If $i : A \to X$ is a cofibration, then i is a injection. [X is Hausdorff, i is closed $\Rightarrow$ embedding]

Lemma: Suppose $\{(X_\alpha, A_\alpha)\}$ is a collection of spaces satisfying HEP, then so does
$$\left(\coprod X_\alpha, \coprod A_\alpha\right)$$

Proof: Exercise.

Lemma: Suppose (X, A) satisfies HEP, $f : A \to B$, $Y = X \cup_f B$

$$\begin{bmatrix} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & & \downarrow \\ X & \longrightarrow & Y \end{bmatrix}$$

then (Y, B) satisfies HEP.

Proof: Exercise

Lemma: Suppose $A = X_0 \subseteq \ldots \subseteq X_n \subseteq \ldots$
$$X = \operatorname{colim} X_n$$
If each (X_i, X_{i-1}) satisfies HEP, then so does (X, A).

Proof: Exercise.

Prop: A relative CW-complex (X, A) satisfies HEP.

Proof: (S^{n-1}, D^n) satisfies HEP

$\Rightarrow (\coprod S^{n-1}, \coprod D^n)$ satisfies HEP

Proof of the prop is by induction. Can inductively assume (X_{n-1}, A) satisfies HEP.

$$\begin{tikzcd} \coprod S^{n-1} \arrow[r] \arrow[d] & X_{n-1} \arrow[d, hook] \\ \coprod D^n \arrow[r] & X_n \end{tikzcd}$$

By the exercise (X_n, A) satisfies HEP.

Taking colimits, (X, A) satisfies HEP.

Fibrations & the HLP

Def^n: A map $p : E \longrightarrow B$ has the $\mathcal{E}$-homotopy lifting property, if for every $A \in \mathcal{E}$, we can complete the diagram:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
A \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & B
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
A \arrow[drr, bend left, "f"] \arrow[dr, dashed, "\tilde{h}"] \arrow[ddr, bend right, "h"'] & & \\
& E^I \arrow[r, "ev_0"] \arrow[d, "p_*"] & E \arrow[d, "p"] \\
& B^I \arrow[r, "ev_0"'] & B
\end{tikzcd}
$$

Rem: $\tilde{H}$ need not be unique.

Def^n: If $\mathcal{C} = \mathrm{Top}$ a map $p: E \to B$ with HLP is called a $\left.\right\}$ fibration.
Hurewicz
If $\mathcal{C} = \{I^n\}$ $= \{\text{CW-complexes}\}$, then this is called a Serre fibration.

Let $E_{p,0} = \{(e, \gamma) \mid \gamma(0) = p(e)\} \subset E \times B^I$

i.e., the pull-back

$$
\begin{tikzcd}
E_{p,0} \arrow[r, "\pi_{B^I}"] \arrow[d, "\pi_E"'] \arrow[dr, phantom, "\lrcorner", very near start] & B^I \arrow[d, "ev_0"] \\
E \arrow[r, "p"'] & B
\end{tikzcd}
$$

Prop: A map $p: E \longrightarrow B$ is a fibration if and only if there exists a map $E_{p,0} \longrightarrow E^I$ making the diagram commute:

$$\begin{tikzcd}
E_{p,0} \arrow[dr, dashed] \arrow[drr, bend left, "\pi_E"] \arrow[ddr, bend right, "\pi_{B^I}"'] & & \\
& E^I \arrow[r] \arrow[d, "p_*"] & E \arrow[d, "p"] \\
& B^I \arrow[r, "ev_0"'] & B
\end{tikzcd}$$

Proof:
- If p is a fibration, this is just HLP
- If there is such a map, use the universal property of the pull-back.

Rem: Fibrations are preserved under pull-back.

Exercise: A composition of fibrations is a fibration.

HELP (!!)

<u>Theorem</u>: Let (X, A) be a relative CW complex of dimension $\leq n$, & let $e: Y \to Z$ be an n-equivalence. Then, given maps

$$f: X \longrightarrow Z$$
$$g: A \longrightarrow Y$$
$$h: A \times I \longrightarrow Z$$

such that

$$f|_A = h \circ i_0$$
$$e \circ g = h \circ i_1$$

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd] & & A \times I \arrow[dl, "h"'] \arrow[dd] & & A \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd] \\
& Z & & Y \arrow[ll, "e"'] & \\
X \arrow[ur, "f"] \arrow[rr, "i_0"'] & & X \times I \arrow[ul, dashed, "\tilde{h}"'] & & X \arrow[ll, "i_1"] \arrow[ul, dashed, "\tilde{g}"']
\end{tikzcd}$$

Rem: Equivalently:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "g"'] & X \arrow[d, "f"] \arrow[dl, dashed, "\tilde{g}"'] \\
Y \arrow[r, "e"'] & Z
\end{tikzcd}$$

Suppose that this square commutes up to homotopy (called H), then $\exists \hat{g}$ such that the upper triangle commutes, & the bottom triangle commutes up to a homotopy $\tilde{H}$ that extends H ($\tilde{H}|_{A\times I} = H$).

Rem 2: Passing to colimits, I can take $n=\infty$.