Complex K-theory as a representable functor

Recall $BU(n) := Gr_n(\mathbb{C}^\infty)$,

$$\mathrm{Vect}^n_{\mathbb{C}}(X) \cong [X, BU(n)]$$

I have a bit lazy with based vs unbased spaces & maps. Let us recall that Top_* denotes the category of pointed topological spaces. The forgetful functor

$$U : Top_* \longrightarrow Top$$

has a left adjoint given by adding a disjoint basepoint. Then

$$\mathrm{Vect}^n_{\mathbb{C}}(X) \cong \underbrace{[X_+, BU(n)]_*}_{\text{base-point-preserving}}$$

We have maps

$$i_n : BU(n) \longrightarrow BU(n+1)$$

& we define $BU := \operatorname{colim} BU(n)$ with the direct limit topology.

<u>Thm</u> For any compact space X

$$(1) \quad K(X) \cong [X_+, BU \times \mathbb{Z}]_*$$

where $\mathbb{Z}$ is given the discrete topology.

If X is a non-degenerately based compact space (i.e, the inclusion of the base point in X is a cofibration), then

$$(2) \quad \tilde{K}(X) \cong [X, BU \times \mathbb{Z}]_* .$$

<u>Proof</u>: We prove (1) first. First note that we can assume X is connected, because both $K(-)$, $[-, BU \times \mathbb{Z}]_*$ send disjoint unions to cartesian products.

Let X be connected, & ξ an n-dim bundle over X with classifying map

$$f_\xi : X \longrightarrow BU(n) \subset BU$$

Every element of $K(X)$ is given as

$$[\xi] - \underbrace{\tau^q}_{\substack{\text{trivial } q\text{-dimensional} \\ \text{bundle over } X}}$$

Then the isomorphism (1) is given by sending

$$[\xi] - \tau^q \longmapsto (f_\xi, n-q)$$

To get (2) we use (1).

Let $S^0 \longrightarrow X_+$ be the cofibration be the basepoint & a disjoint base point.

Last week we used the map $d: K(X) \longrightarrow K(*) \cong \mathbb{Z}$ to define reduced K-theory. This map d can be identified with

$$[X_+, BO\times\mathbb{Z}]_* \longrightarrow [S^0, BO\times\mathbb{Z}]_*$$

& so we must show that the kernel

is $[X, BO\times\mathbb{Z}]_*$.

The cofibration

$$S^0 \longrightarrow X_+ \longrightarrow X$$

induces

$$[S^1\wedge S^0, BO\times\mathbb{Z}]_* \longrightarrow [X, BU\times\mathbb{Z}] \longrightarrow [X_+, BO\times\mathbb{Z}] \longrightarrow [S^0, BO\times\mathbb{Z}]$$

$$\begin{aligned} &\;\;\shortparallel \\ &[S^1, BO\times\mathbb{Z}]_* \\ &\;\;\shortparallel \\ &[S^1, BO]_* \\ &\;\;\shortparallel \\ &\pi_1(BO) = \pi_0(O) = 0. \end{aligned}$$

<u>Def</u>: If X is a space with the homotopy type of a CW-complex, then

$$K(X) := [X_+, BO\times\mathbb{Z}]_*$$

If X is a non-degenerately based space with the homotopy type of a CW complex, then

$$\tilde{K}(X) := [X, BO\times\mathbb{Z}]_*.$$

Rem: there are other possible definitions. For example, I could write X as the limit of it's finite CW-skeletons, & take $\mathbb{K}(X) := \varprojlim K(X^n)$

These don't always agree.

Prop: the space $BU \times \mathbb{Z}$ is a ring up to homotopy. In particular, $K(X)$ is a ring.

Proof sketch: Taking direct sums induces

$$Gr_m(\mathbb{C}^\infty) \times Gr_n(\mathbb{C}^\infty) \longrightarrow Gr_{n+m}(\mathbb{C}^\infty \oplus \mathbb{C}^\infty)$$

Choosing an isomorphism $\mathbb{C}^\infty \oplus \mathbb{C}^\infty \cong \mathbb{C}^\infty$ we get a map

$$BU(m) \times BU(n) \longrightarrow BU(m+n)$$

Taking colimits gives

$$\oplus : BU \times BU \longrightarrow BU$$

This map is associative & commutative up to homotopy. Using the addition on $\mathbb{Z}$, we extend this to $BO \times \mathbb{Z}$.

For multiplication, we start with the tensor product of canonical bundles, & show this induces

$$BO(n) \times BO(m) \longrightarrow BO(n+m)$$

With some effort show these are compatible with the colimits, & so induce

$$\otimes : BO \times BO \longrightarrow BO \qquad \square.$$

K-theory as a cohomology theory

In ordinary cohomology we have a graded ring structure $H^i(X) \smile H^j(X) \in H^{i+j}(X)$

the same is true in K-theory.

If X is a pointed space, then for $i, j \geq 0$ there are maps $[S^{i+j} \cong S^i \wedge S^j]$

$$S^{i+j} \wedge d : S^{i+j} \wedge X \longrightarrow S^{i+j} \wedge X \wedge X$$
$$\cong (S^i \wedge X) \wedge (S^j \wedge X)$$

& so maps

$$\tilde{K}(S^i \wedge X) \otimes \tilde{K}(S^j \wedge X) \xrightarrow{- \boxtimes -} \tilde{K}(S^i \wedge X \wedge S^j \wedge X)$$
$$\xrightarrow{d^*} \tilde{K}(S^{i+j} \wedge X)$$

i.e. a map

$$- \otimes - : \tilde{K}^{-i}(X) \otimes \tilde{K}^{-j}(X) \longrightarrow \tilde{K}^{-i-j}(X)$$

Using Bott-periodicity this extends to a multiplication defined on $i, j \in \mathbb{Z}$.

<u>Rem</u>: This multiplication is graded-commutative.

Unreduced K-theory

Note that $\tilde{K}^0(X_+) = K(X)$, so we just define

$$K^i(X) := \tilde{K}^i(X_+)$$

So, for example, $K^{-1}(*) \cong \tilde{K}^{-1}(*_+)$

$$\cong \tilde{K}^{-1}(S^0)$$

$$= 0$$

, as we saw on Friday.

With this definition, we get an exact sequence

$$\begin{tikzcd}
K^0(A) \arrow[d, "\partial"'] & K^0(X) \arrow[l, "i^*"'] & \tilde{K}^0(X/A) \arrow[l, "q^*"'] \\
\tilde{K}^{-1}(X/A) \arrow[r, "q^*"] & K^{-1}(X) \arrow[r] & K^{-1}(A) \arrow[u, "\partial"']
\end{tikzcd}$$

Mayer-Vietoris sequence

X is a compact Hausdorff space

$X = A \cup B$ two closed subspaces $A, B \subset X$.

Then $X/A \cong B/(A \cap B)$,

so there is a map of long exact sequences

$$\begin{tikzcd}
\tilde{K}^{-1}(X/A) \arrow[d, "\cong"] & K^0(A) \arrow[l, "\delta"'] \arrow[d, "\hat{j}_A^*"'] & K^0(X) \arrow[l, "\hat{i}_A^*"'] \arrow[d] & \tilde{K}^0(X/A) \arrow[l, "q_X^*"'] \arrow[d, "\cong"'] & K^{-1}(A) \arrow[l, "\delta"'] \arrow[d] \\
\tilde{K}^{-1}\left(B/A\cap B\right) & K^0(A \cap B) \arrow[l, "\delta"] & K^0(B) \arrow[l, "\hat{j}_B^*"] & \tilde{K}^0\left(B/A\cap B\right) \arrow[l, "q_{A\cap B}^*"] & K^{-1}(A \cap B) \arrow[l, "\delta"']
\end{tikzcd}$$

Exercise: Suppose I have a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C''_{n+1} \arrow[r, "\delta_{n-1}"] \arrow[d, "f''_{n+1}"', "\cong"] & C'_n \arrow[r, "p_n"] \arrow[d, "f'_n"'] & C_n \arrow[r, "i_n"] \arrow[d, "f_n"'] & C''_n \arrow[r, "\delta_n"] \arrow[d, "f''_n"', "\cong"] & C'_{n+1} \arrow[d, "f'_{n+1}"'] \\
\cdots \arrow[r] & D''_{n+1} \arrow[r, "\delta'_{n-1}"] & D'_n \arrow[r, "p'_n"'] & D_n \arrow[r, "i'_n"'] & D''_n \arrow[r] & C'_{n+2}
\end{tikzcd}$$

where the rows are exact, & f''_n are

isomorphisms. Then there is a long exact sequence

$$\cdots \longrightarrow C_n \xrightarrow{(i_n, f_n)} C_n' \oplus D_n \xrightarrow{f_n'' - \partial_n'} D_n'' \xrightarrow{\Delta_n} C_{n+1} \longrightarrow \cdots$$

where $\Delta_n = p_{n+1} \circ (f_{n+1})^{-1} \circ \delta_n'$

<u>Cor</u>: (Mayer-Vietoris in K-theory)

There is a long exact sequence

$$\begin{tikzcd}
K^0(A \cap B) \arrow[d, "\delta'"'] & K^0(A) \oplus K^0(B) \arrow[l, "j_A^* - j_B^*"'] & K^0(X) \arrow[l, "i_A^* \oplus i_B^*"'] \\
K^{-1}(X) \arrow[r, "i_A^* \oplus i_B^*"] & K^{-1}(A) \oplus K^{-1}(B) \arrow[r, "j_A^* - j_B^*"] & K^{-1}(A \cap B) \arrow[u]
\end{tikzcd}$$

Adams operations & Hopf invariant one

<u>Def^n</u>: An operation Θ in K-theory assigns a function $\Theta_X : K(X) \longrightarrow K(X)$ in such a way that the diagram

$$\begin{tikzcd} K(Y) \arrow[r, "\otimes_Y"] \arrow[d, "f^*"'] & K(Y) \arrow[d, "f^*"] \\ K(X) \arrow[r, "\otimes_X"'] & K(X) \end{tikzcd}$$

is commutative.

<u>Theorem</u>: For each non-zero integer k, & each compact Hausdorff space X, there is a ring homomorphism

$$\psi^k : K(X) \longrightarrow K(X)$$

satisfying the following

(1) $\psi^1 = $ identity, & ψ^{-1} is induced by conjugation of complex bundles.

(2) $\psi^k f^* = f^* \psi^k$ for all map $f : X \longrightarrow Y$

(3) $\psi^k(L) = L^k = L \otimes \cdots \otimes L$ if L is a line bundle

(4) $\psi^k \circ \psi^\ell = \psi^{k\ell}$

(3) $\psi^p(\alpha) \equiv \alpha^p \mod p$ for p a prime,

i.e. $\psi^p(\alpha) - \alpha^p = p\beta$ for some $\beta \in K(X)$.

Rem: By (4) $\psi^{-k} = \psi^k \circ \psi^{-1}$

If we use (1) to define ψ^{-1}, then we need only construct ψ^k for $k > 1$.

Exterior powers

Recall that the exterior power of a vector space is $V \otimes \dots \otimes V$ modulo the subspace generated $v_1 \otimes \dots \otimes v_k - \operatorname{sgn}(\sigma)\, v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)}$

where σ is a permutation, $\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{even per} \\ 0 & \text{odd per} \end{cases}$

Extending this to bundles, we can form an exterior power $\lambda^k(E)$ with the following properties:

(i) $\lambda^k(E_1 \oplus E_2) = \bigoplus_{i+j=k} \lambda^i(E_1) \oplus \lambda^j(E_2)$

(ii) $\lambda^0(E) = 1$, the trivial bundle.

(iii) $\lambda^1(E) = E$

(iv) $\lambda^k(E) = 0$ for $k >$ max dimension of the fibes of E.

<u>Lemma</u>: the λ^k extend to operations on K-theory $\lambda^k : K(X) \longrightarrow K(X)$.

<u>Proof</u>: Consider the ring $K(X)[[t]]$ of formal power series, & let G the multiplicative group of units in $K(X)[[t]]$ (formal power series whose leading coefficient is 1).

We define a function

$$\lambda : \mathrm{Vect}(X) \longrightarrow G$$

by setting

$$\lambda(E) = 1 + \lambda^1(E)t + \lambda^2(E)t^2 + \dots + \lambda^k(E)t^k + \dots$$

Property (i) tells us that

$$\lambda(E_1 \oplus E_2) = \lambda(E_1)\cdot\lambda(E_2)$$

So we have a homomorphism of commutative monoids

$$\mathrm{Vect}(X) \longrightarrow G$$

By the universal property of Grothendieck completion

$$\begin{tikzcd} \mathrm{Vect}(X) \arrow[r, "i"] \arrow[dr] & K(X) \arrow[d, dashed, "\Lambda_t\ \exists!"] \\ & G \end{tikzcd}$$

We define $\lambda^k(x)$ to be the coefficient of t^k in $\Lambda(x)$. $\square$.

Returning to Adams operations, suppose $E = L_1 \oplus \dots \oplus L_n$ which is a sum of line bundles.

Then,

$$\psi^k(E) = \psi^k(L_1 \oplus \dots \oplus L_n)$$

$$= \psi^k(L_1) \oplus \dots \oplus \psi^k(L_n)$$

$$= L_1^k \oplus \dots \oplus L_n^k$$

The construction of ψ^k will be based on the construction of a polynomial Q_k with integral coefficients for which

$$L_1^k \oplus \dots \oplus L_n^k = Q_k(\lambda^1(E), \dots, \lambda^k(E))$$

We will then define

$$\psi^k(E) := Q_k(\lambda^1(E), \dots, \lambda^k(E)).$$