Characteristic classes

We saw in the last lecture that $BU(n)$ classifies rank n complex vector bundles.

Prop $H^*(BU(n);\mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_n]$

$|c_i| = 2i$

For $H < G$ a normal subgroup there is a fibration

$$G/H \longrightarrow BH \longrightarrow BG$$

Moreover $U(n)/U(n-1) \simeq S^{2n-1}$

ie, there is a fibration

$$S^{2n-1} \longrightarrow BU(n-1) \longrightarrow BU(n)$$

Using that $BU(1) \simeq \mathbb{C}P^\infty$, this starts an inductive argument to get the result.

Another way to do it is to recall that $H^*(U(n);\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[x_1, x_3, \dots, x_{2n-1}]$

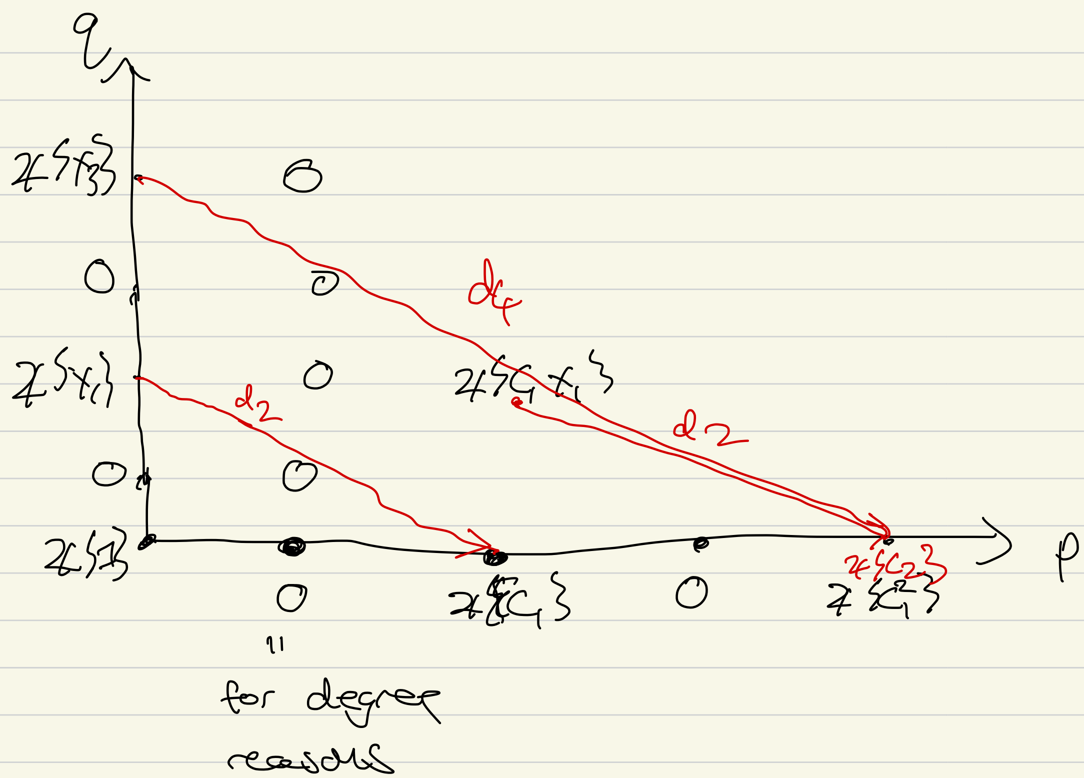
We have the universal bundle

$$U(n) \longrightarrow EU(n) \longrightarrow BU(n)$$

The Serre spectral sequence takes the form

$$E_2^{p,q} = H^p(BU(n); H^q(U(n))) \Longrightarrow H^{p+q}(EU(n)) = \begin{cases} \mathbb{Z} & p+q=0 \\ 0 & \text{else} \end{cases}$$

$$\cong H^p(BU(n)) \otimes H^q(U(n))$$



$\square$.

Def^n: the generators $c_1, \dots, c_n$ are $H^* BU(n)$ are called the universal **Chern classes** of $U(n)$-bundles.

Given $\pi : E \longrightarrow X$ a principal $U(n)$ bundle, there exists a classifying map $f_\pi : X \longrightarrow BU(n)$, such that

$\pi \cong f_\pi^* \pi_{U(n)}$.

Def^n: The i-th Chern class of the $U(n)$-bundle $\pi : E \longrightarrow X$ with classifying map $f_\pi : X \longrightarrow BU(n)$ is defined as

$$c_i(\pi) := f_\pi^*(c_i) \in H^{2i}(X; \mathbb{Z})$$

Rem: If π is a $U(n)$-bundle, then $c_i(\pi) = 0$ for $i > n$.

Prop: If ε denotes the trivial $U(n)$-bundle on a space X, then $c_i(\varepsilon) = 0$ for all $i > 0$.

Proof: The classifying map for the trivial bundle is the constant map $ct : X \longrightarrow BU(n)$, which induces the trivial homomorphism in positive degree cohomology.

Prop (functoriality) If $f: Y \longrightarrow X$ is a cts map & $\pi : E \longrightarrow X$ is a $U(n)$-bundle, then $c_i(f^*\pi) = f^* c_i(\pi)$ for any i.

Rem: f^* has two different meanings here once as a pullback-bundle, & one as the induced map in cohomology.

Def^n: The total Chern class of a $U(n)$-bundle $\pi: E \longrightarrow X$ is defined by

$$c(\pi) = c_0(\pi) + c_1(\pi) + \dots + c_n(\pi)$$
$$= 1 + c_1(\pi) + \dots + c_n(\pi)$$

Def^n (Whitney sum) π_1 a principal $U(n)$-bundle over X

π_2 a principal $U(m)$ bundle over X

$\pi_1 \times \pi_2$ is a principa $U(n) \times U(m)$-bundle.

$$U(n) \times U(m) \longmapsto U(n+m)$$
$$(A, B) \longmapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

$\pi_1 \times \pi_2$ can be considered as a $U(n+m)$-bundle.

The Whitney sum is

$$\pi_1 \oplus \pi_2 = \Delta^* (\pi_1 \times \pi_2)$$

where

$$\begin{aligned} \Delta : X &\longrightarrow X \times X \\ a &\longmapsto (a, a) \end{aligned}$$

$\pi_1 \oplus \pi_2$ is a $U(n+m)$-bundle over X.

Prop: With π_1, π_2 as above

$$c(\pi_1 \oplus \pi_2) = c(\pi_1) \cdot c(\pi_2)$$

the cup product of $c(\pi_1)$ & $c(\pi_2)$

Equivalently,

$$c_k(\pi_1 \oplus \pi_2) = \sum_{i+j=k} c_i(\pi_1) \cdot c_j(\pi_2)$$

Proof: We always have $B(G \times H) \simeq BG \times BH$. (exercise for this week). In particular,

$$B(U(n) \times U(m)) \simeq BU(n) \times BU(m).$$

The inclusion

$$U(n) \times U(m) \longrightarrow U(n+m)$$

yields

$$\omega : \underset{\substack{\cong \\ BU(n) \times BU(m)}}{B(U(n) \times U(m))} \longrightarrow BU(n+m)$$

<u>Claim</u>: $\omega^*(c_k) = \sum_{i+j=k} c_i \otimes c_j \qquad [c_0 = 1]$

Assuming the claim for a moment, & noting that the classifying map for $\pi_1 \times \pi_2$ regarded as a $U(n+m)$-bundle is $\omega \circ (f_1 \times f_2)$, we get isomorphism

$$\begin{aligned}
c_k(\pi_1 \oplus \pi_2) &= c_k(\Delta^*(\pi_1 \times \pi_2)) \\
&= \Delta^* c_k(\pi_1 \times \pi_2) \\
&= \Delta^*(f^*_{\pi_1 \times \pi_2}(c_k)) \\
&= \Delta^*(f^*_{\pi_1} \times f^*_{\pi_2}(\omega^* c_k)) \\
&= \sum_{i+j=k} \Delta^*(f^*_{\pi_1}(c_i) \otimes f^*_{\pi_2}(c_j)) \\
&= \sum_{i+j=k} \Delta^*(c_i(\pi_1) \otimes c_j(\pi_2))
\end{aligned}$$

$$= \sum_{i+j=n} c_i(\pi_1) \smile c_j(\pi_2).$$ $\square$

$$H^i \smile H^j \longrightarrow H^{i+j}$$

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the claim is a little tricky. Here is a sketch:

there is a canonical map $U(1)^{\times n} \longrightarrow U(n)$

this gives a diagram

$$\begin{tikzcd}
c_t \in H^* BU(n) \arrow[r, "\mu_{k,n-k}^*"] \arrow[d, "\mu_n^*"'] \arrow[dr, phantom, "\circlearrowright"] & H^*(BU(k)) \otimes H^*(BU(n-k)) \arrow[d, "\mu_k^* \otimes \mu_{n-k}^*"] \\
H^*(BU(1)^{\times n}) \arrow[r, "\cong"] & H^*(BU(1)^k) \otimes H^*(BU(1)^{n-k})
\end{tikzcd}$$

$$(\mu_k^* \otimes \mu_{n-k}^*) \circ \mu_{k,n-k}^*(c_t) = \mu_n^*(c_t)$$

The computational fact we need is that

$$BU(1)^n \longrightarrow BU(n)$$

induces a map

$$\mathbb{Z}[c_1,\dots,c_n] \longrightarrow \mathbb{Z}[(c_1)_1,\dots,(c_1)_n]$$

$$c_k \longmapsto \underbrace{\sigma_k((c_1)_1,\dots,(c_1)_n)}$$

$\sigma_1(\dots) = (c_1)_1 + \dots + (c_1)_n$

$\sigma_2(\dots) = (c_1)_1(c_1)_2 + (c_1)_1(c_1)_3 + \dots$

$\vdots$

$\sigma_n(\dots) = (c_1)_1(c_1)_2\cdots(c_1)_n$

k-th elementary sym. polynomial in n-copies of the first Chern class

$$\mu_n^*(c_t) = \sigma_t((c_1)_1,\dots,(c_1)_n)$$

$$= \sum_{r=0}^{t}\sigma_r((c_1)_1,\dots,(c_1)_{n-t})\cdot\sigma_{t-r}((c_1)_{n-t-1},\dots,(c_1)_n)$$

$$(\mu_k^*\otimes\mu_{n-k}^*)(\mu_{k,n-k}^*(c_k)) = (\mu_k^*\otimes\mu_{n-k}^*)\left(\sum_{r=0}^{t}c_r\otimes c_{t-r}\right)$$

$\mu_k^*\otimes\mu_{n-k}^*$ is a monomorphism

$$\Rightarrow \mu_{k,n-k}^*(c_k) = \sum_{r=0}^{t}c_r\otimes c_{t-r}$$

Cor: Let ε^1 be the trivial $U(1)$-bundle, then $c(\pi\oplus\varepsilon^1) = c(\pi)$

Example: $c_1(\gamma_1^1)$ first Chern class of the rank 1 tautological bundle over $\mathbb{C}P^1$.

$$\begin{tikzcd} \gamma_1^1 \arrow[r] \arrow[d] & S^\infty \arrow[d, "q"] \\ \mathbb{C}P^1 \arrow[r, "f_\pi"'] & \mathbb{C}P^\infty \end{tikzcd}$$

$$H^*(\mathbb{C}P^\infty) \longrightarrow H^*(\mathbb{C}P^1)$$
$$\mathbb{Z}[t] \longmapsto \frac{\mathbb{Z}[t]}{(t^2)}$$

What is f_π^*?

I claim that f_π is just the inclusion map, & so $c_1(\gamma_1^1)$ is the canonical degree 2 general of $H^*(\mathbb{C}P^\infty)$.

More generally, consider the inclusion map

$$i : \mathbb{C}P^n \hookrightarrow \mathbb{C}P^\infty$$

then the total space of the pull-back bundle is

$$\{([e], x) \in \mathbb{C}P^n \times S^\infty \mid i([e]) = q(x)\}$$
$$= \{([e], x) \in \mathbb{C}P^n \times S^{2n-1} \mid i([e]) = q(x)\}$$
$$= \{([e], x) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} \mid x \in [e]\}$$

$= \gamma_n^1$.

In fact, Chern classes are completely determined by the following axioms.

(C1) $c_0(\pi) = 1$ & $c_k(\pi) = 0$ for $k > \mathrm{rank}(\pi)$

(C2) For a cts map f, $c(f^*\pi) = f^*(c(\pi))$

(C3) (Whitney sum formula)

$$c(\pi_1 \oplus \pi_2) = c(\pi_1) \cdot c(\pi_2)$$

(C4) $c_1(\gamma_1^1)$ is isomorphic to the canonical degree 2 generator of $H^*(\mathbb{C}P^\infty)$.

How do we prove this?

Let $\pi : E(\pi) \longrightarrow B$ be an n-dimensional complex vector bundle. The group $\mathbb{C}^\times$ act fiberwise on non-zero elements of $E(\pi)$ by scalar multiplication. Let $P(\pi)$ be the orbit space. The projection map $\pi : E(\pi) \longrightarrow B$ induces a projection $p_\pi : P(\pi) \longrightarrow B$. This is called the projective bundle associated to π. The fiber $p_\pi^{-1}(b)$ is the projective space $P(\pi_b) \cong \mathbb{R}P^{n-1}$ of the vector space $\pi^{-1}(b)$.

<u>Thm</u> $p_\pi^* : H^* B \longrightarrow H^* P(\pi)$ is an injection. (Leray-Hirsch theorem)

<u>Prop</u>: For each complex vector bundle of rank n $\pi : E \longrightarrow B$ there exists a space $F(E)$ & a map $p : F(E) \longrightarrow B$ such that

$$\begin{array}{c} p^* E \\ \downarrow \\ F(E) \end{array}$$

splits as a direct sum of line bundles & $p^* : H^* B \longrightarrow H^* F(E)$ is injective.

Proof: Consider the pull-back along $p(\pi): P(E) \longrightarrow B$. Call this bundle $p(\pi)^*(E)$. This pull-back contains a natural 1-dimensional sub-bundle

$$L = \{(e, v) \in P(E) \times E \mid v \in e\}$$

then

$$p(\pi)^*(E) = L \oplus L^{\perp} \quad \text{where } \dim(L^{\perp}) = n-1$$

Now iterate this procedure using $L^{\perp} \longrightarrow P(E)$ instead of $E \longrightarrow B$.

Thm the (4) axioms completely determine the Chern classes.

Proof: Let c_i & $\overline{c_i}$ be two sets of Chern classes. Let $\pi: E \longrightarrow B$ be an n-dimensional vector-bundle, with splitting $p: F(E) \longrightarrow B$ (i.e, $p^*\pi = \lambda_1 \oplus \dots \oplus \lambda_n$)

Then

$$p^{\sigma}(c(\pi)) = c(p^{\sigma}\pi)$$
$$= c(\lambda_1 \oplus \cdots \oplus \lambda_n)$$
$$= c(\lambda_1)\cdot c(\lambda_2)\cdots\cdot c(\lambda_n)$$
$$= (1+c_1(\lambda_1))(1+c_1(\lambda_2))\cdots(1+c_1(\lambda_n))$$
$$= (1+\bar{c}_1(\lambda_1))(1+\bar{c}_1(\lambda_2))\cdots(1+\bar{c}_1(\lambda_n))$$
$$= \bar{c}(p^{\sigma}\pi)$$
$$= p^{\sigma}(\bar{c}(\pi))$$

But p^{σ} is injective, $\Rightarrow$ $c(\pi) = \bar{c}(\pi)$ $\square$.