Week 10

Exercise: Suppose I have a morphism of fiber bundles

$$\begin{tikzcd} E' \arrow[r, "\varphi'"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ B' \arrow[r, "\varphi"] & B \end{tikzcd}.$$

Then, $\pi' \cong \varphi^* \pi$.

Associated bundle

Let (π, P, B) be a principal G-bundle, & let $e: G \to \mathrm{Aut}(F)$ a left action on a space F. Then the product $P \times F$ has a canonical right action

$$(p, f) \cdot g = (p \cdot g, \underbrace{e(g^{-1}) \cdot f}_{g^{-1} \cdot f})$$

Prop: the quotient of $P \times F$ by G defines a fiber bundle over B with fiber F by

$$P \times_G F := (P \times F)/G \xrightarrow{\pi_F} B$$
$$[p, f] \longmapsto \pi(p)$$

This is called the associated fiber bundle. If P is trivial as a G-bundle, then $P \times_G F$ is trivial for any F.

<u>Proof</u>: We first check that π_F is well-defined. Suppose that $[p', f']$ is another representative in this equivalence class. Then $p' = p \cdot g$ for some $g \in G$
$$f' = g^{-1} \cdot f$$

$$\begin{aligned}
\pi_F([p', f']) &= \pi_F([p \cdot g, g^{-1} \cdot f]) \\
&= \pi(p \cdot g) \\
&= \pi(p) \\
&= \pi_F([p, f])
\end{aligned}$$

We claim that the fibers of $P \times_G F$ are homeomorphic to F. To see this, fix a point $b \in B$, & choose a point $p_0 \in \pi^{-1}(b)$ in the fiber of P over B. We define a cts map

$$\begin{aligned} F &\longrightarrow \pi_F^{-1}(b) \\ f &\longmapsto [p_0, f] \end{aligned}$$

This has a cts inverse given by

$$\begin{aligned} \pi_P^{-1}(b) &\longrightarrow F \\ [p, f] &\longmapsto \tau(p_0, p) \cdot f \end{aligned}$$

where $\tau(p_0, p) \in G$ is the unique element such that $p_0 \cdot \tau(p_0, p) = p$.

$$[p, f] \longmapsto [p, \underbrace{\tau(p_0, p_0)}_{e} \cdot f]$$

The map $\pi^{-1}(B) \times F \ni (p, f) \longmapsto \tau(p_0, p) \cdot f \in F$

is invariant with respect to the

G-action

$$(p, f) \cdot g = (p \cdot g, \; g^{-1} \cdot f)$$

$$\longmapsto \tau(p_0, p \cdot g) \, g^{-1} \cdot f$$

$$= \tau(p_0, p) \cdot g \cdot g^{-1} \cdot f$$

$$= \tau(p_0, p) \cdot f$$

& hence this descends to the quotient

$$\pi^{-1}(B) \times F / G = \pi_F^{-1}(B).$$

To finish the proof it suffices to prove the final claim: triviality of P implies triviality of $P \times_G F$.

Hence we can assume $P = B \times G$.

Then $(P \times F)/G = (B \times G \times F)/G$

This is isomorphic to $B \times F$

$$([b, g, f]) \longmapsto (b, g\cdot f)$$

with inverse

$$(b, f) \longmapsto [(b, e, f)]. \qquad \square.$$

Remark (structure group of a bundle)

Suppose B is covered by a collection of sets $\{U_\alpha\}$ with local trivializations

$$\varphi_\alpha : U_\alpha \times F \xrightarrow{\cong} p^{-1}(U_\alpha)$$

Let's compare the local trivializations on the intersection $U_\alpha \cap U_\beta$.

$$\begin{tikzcd} U_\alpha \cap U_\beta \times F \arrow[r, "\cong", "\varphi_\beta"'] \arrow[dr, dashed] & p^{-1}(U_\alpha \cap U_\beta) \arrow[d, "\cong"', "\varphi_\alpha^{-1}"] \\ & U_\alpha \cap U_\beta \times F \end{tikzcd}$$

For every $x \in U_\alpha \cap U_\beta$, we obtain a homeomorphism of the fiber F, a map

$$\varphi_{\alpha,\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Aut}(F)$$

these are called <u>clutching functions</u>.

If the bundle is an n-dimensional vector-bundle, then the clutching functions take the form

$$\varphi_{\alpha,\beta} : U_\alpha \cap U_\beta \longrightarrow GL(n, \mathbb{F})$$

If $p : E \longrightarrow B$ is a principal G-bundle, then the clutching functions take values in G:

$$\varphi_{\alpha,\beta} : U_\alpha \cap U_\beta \longrightarrow G \subset \mathrm{Aut}(F).$$

In general, a fiber bundle with structure group G is one whose clutching functions take values in G.

Prop: Given any fiber bundle $\pi: E \longrightarrow B$ with fiber F & structure group $\mathrm{Aut}(F)$, there exists a principal $G := \mathrm{Aut}(F)$-bundle P such that $E = P \times_G F$.

Proof: For $b \in B$, define

$$\underbrace{P_b = \{ G\text{-isomorphisms } \varphi : F \longrightarrow \pi^{-1}(b) \}}_{\text{set of frames}}$$

This has an action of G on the right

$$\varphi \cdot g : F \longrightarrow \pi^{-1}(b)$$

is also a G-isomorphism

This is a free & transitive action:

any two $\varphi, \varphi' : F \longrightarrow \pi^{-1}(b)$

are related by $g = \varphi'^{-1} \circ \varphi \in G = \mathrm{Aut}(F)$.

Define

$$P = \bigcup_{b \in B} P_b$$

& define

$$\begin{aligned} \pi_P : P &\longrightarrow B \\ p \in P_b &\longmapsto b \end{aligned}$$

Suppose $E = B \times F$, then $\pi^{-1}(b) = \{b\} \times F$

& canonically, $P_b \cong G$. Thus in this case

$$P = \bigcup_{b \in B} P_b = \bigcup_{b \in B} \{b\} \times G$$

$$= B \times G$$

In the general case we do the same construction over local trivializations.

To see that $P \times_G F \cong E$, note that points in $P \times_G F$ are equivalence classes $[b, \varphi, f]$ where $b \in B$, $\varphi : F \longrightarrow \pi^{-1}(b)$ is a G-isomorphism & $f \in F$.

Now consider

$$P \times_G F \ni [b, \varphi, f] \longmapsto \varphi(f) \in E.$$

This is well-defined

$$[b, \varphi\cdot g, g^{-1}\cdot f] \longmapsto \varphi(g\cdot g^{-1} f)$$
$$= \varphi(f)$$

Because φ is an isomorphism, this gives the bundle isomorphism. $\square$.

Notation: F is a space with a left G-action, a cts map $f : P \longrightarrow F$ is G-equivariant if $f(p\cdot g) = g^{-1}\cdot f(p)$. We denote the set of such maps by $Map_G(P, F)$.

Prop: Let (π, P, B) be a principal G-bundle, F a space with a left G-action, & $E = P \times_G F$ the associated bundle. Then there is a bijection

$$\Gamma(U, E) \rightleftarrows \mathrm{Map}_G(\pi^{-1}(U), F).$$

Proof: Given $\tilde{s} \in \mathrm{Map}_G(\pi^{-1}(U), F)$ define

$$s(b) := [p, \tilde{s}(p)]$$

This is well-defined

$$[p \cdot g, \tilde{s}(p \cdot g)] = [p \cdot g, g^{-1} \tilde{s}(p)]$$
$$= [p, \tilde{s}(p)].$$

Conversely, let $s : U \longrightarrow E$ be a local section. Suppose $s(\pi(p)) = [p, f]$

Define

$$\tilde{s} : \pi^{-1}(U) \longrightarrow F$$
$$p \longmapsto f$$

To see this is G-equivariant

$$\tilde{s}(p \cdot g) = g^{-1} \cdot \tilde{s}(p)$$

because
$$\begin{aligned} s(\pi(p \cdot g)) &= s(\pi(p)) \\ &= [p, f] \\ &= [p \cdot g, g^{-1} \cdot f] \end{aligned}$$

From the defintions s & $\tilde{s}$ are inverse bijections. Also, s cts $\Leftrightarrow$ $\tilde{s}$ cts $\square$.

<u>Prop</u>: Fix a group G. Let (π, P, B) & (π', Q, B') be principal G-bundles over B & B', then there is a bijection between bundle morphisms $\varphi : (\pi, P, B) \longrightarrow (\pi', Q, B')$ & sections of the associated

Q is a left G-space with action $g \cdot q = q \cdot g^{-1}$.

bundle $P \times_G Q$

Proof: A morphism as above is specified by a G-map $\phi : P \longrightarrow Q$ i.e. an element $\mathrm{Map}_G(P, Q)$.

$$\mathrm{Map}_G(P, Q) = \mathrm{Map}_G(\pi^{-1}(B), Q)$$

$$[\text{Previous prop} \Rightarrow] = \Gamma(B, P \times_G Q).$$

Homotopy classification of principal bundles

Prop: If (π, P, B') a principal G-bundle, & $f_0 \sim f_1 : B \longrightarrow B'$, then $f_0^*(\pi)$ & $f_1^*(\pi)$ are isomorphic as bundles over B.

Rem: Use the associated bundle

constructions, can prove it holds for any fiber bundle.

Exercise: Fiber bundles over a contractible base space are trivial.

Proof of proposition:

Let $f_t : B \times I \longrightarrow B$ be a homotopy between f_0 & f_1. Now consider the following diagram:

$$\begin{tikzcd}
f_0^* P \arrow[rr] \arrow[dd, "f_0^*\pi"'] & & f_t^* P \arrow[r] \arrow[dd, "f_t^*\pi"] & P \arrow[dd, "\pi"] \\
& f_1^* P \arrow[ur] \arrow[dd, "f_1^*\pi"] & & \\
B \times \{0\} \arrow[rr, hook, "i_0"] & & B \times I \arrow[r, "f_t"] & B \\
& B \times \{1\} \arrow[ur, "i_1"'] & &
\end{tikzcd}$$

Claim: For a principal G-bundle $(\pi, Q, B\times I)$ we have $i_0^*\pi \simeq i_1^*\pi$

$i_0 : B\times\{0\} \longrightarrow B\times I$
$i_1 : B\times\{1\} \longrightarrow B\times I$

This will complete the proof as then

$$f_0^*\pi \simeq i_0^* f_t^*\pi \simeq i_1^* f_t^*\pi \simeq f_1^*\pi$$

Proof of the claim:

$$\begin{tikzcd}
i_0^*\pi \arrow[rr] \arrow[dd, "\pi_0"'] & & Q \arrow[dd, "\pi"] \\
& i_1^*\pi \arrow[ur] \arrow[dd, "\pi_1"'] & \\
B\times\{0\} \arrow[rr, hook, "i_0"'] & & B\times I \\
& B\times\{1\} \arrow[ur, hook, "i_1"'] &
\end{tikzcd}$$

We claim in fact that

$$\pi \cong \pi_0 \times id_I$$

If this holds, then

$$Q\big|_{B\times\{1\}} = i_1^*\pi$$
$$= (Q\times I_0)\big|_{B\times I} \simeq i_0^*\pi$$

Since we work with principal G-bundles morphisms over a fixed base are isomorphisms. Therefore it suffices to produce a morphism

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & Q_0 \times I \arrow[d] \\ B \times I \arrow[r, "id"] & B \times I \end{tikzcd}$$

By the previous proposition, this corresponds to a section of the bundle

$$\omega : Q \times_G (Q_0 \times I) \longrightarrow B \times I.$$

Now, by definition, $Q \times_G (Q_0 \times I)$ has a section s over $B \times \{0\}$. Now consider the following diagram

$Q|_{B \times \{0\}}$ & $Q_0 \times I|_{B \times \{0\}}$ are both just Q_0.

$$\begin{tikzcd} B \times \{0\} \arrow[r, "s"] \arrow[d, "i_0"'] & Q \times_G (G_0 \times I) \arrow[d, "\omega"] \\ B \times I \arrow[r, "\text{identity}"'] \arrow[ur, dashed, "\tilde{s}"] & B \times I \end{tikzcd}$$

We want to find a lift $\tilde{s}$ in this diagram. The lift $\tilde{s}$ exists by the homotopy lifting property, because ω is a fibration. $\square$.

Rem: $\mathcal{G}(B)$ = set of principal G-bundles over B. This is functorial with respect to cts maps $f: A \longrightarrow B$

$$P \longmapsto f^*(P)$$

The previous proposition shows that $\mathcal{G}$ descends to the homotopy category.

$$\mathcal{G} : h\mathrm{Top} \longrightarrow \left\{ \begin{array}{c} \text{principal } G\text{-bundles} \\ \text{up to isomorphism} \end{array} \right\}.$$

Week 10 - Lecture 2

Classification of principal G bundles

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M \arrow[d] \\ S^1 \arrow[r, "z^2"] & S^1 \end{tikzcd}$$

Def^n: $\mathcal{G}(B)$ = set of isomorphism classes of principal G-bundles over B.

$$B \longmapsto \mathcal{G}(B)$$

We have a contravariant functor

$$\mathcal{G} : \mathrm{Top} \longrightarrow \left\{ \begin{array}{c} \text{principal } G\text{-bundles} \\ \text{up to isomorphism} \end{array} \right\}$$

which sends $f : A \longrightarrow B$

to $f^* : \mathcal{G}(B) \longrightarrow \mathcal{G}(A)$

$$P \longmapsto f^* P$$

the proposition from Tuesday shows that this factors through the homotopy category

$$\begin{tikzcd} \mathrm{Top} \arrow[r, "\mathcal{G}"] \arrow[d] & \left\{\begin{array}{c}\text{principal}\\ G\text{-bundles up}\\ \text{to iso}\end{array}\right\} \\ h\mathrm{Top} \arrow[ur, "\mathcal{G}"'] & \end{tikzcd}$$

<u>Rem</u>: For today B is a CW-complex.

<u>Def^n</u>: A principal G-bundle (π_G, EG, BG) is called <u>universal</u> if the

$$\begin{array}{c} EG \\ \pi_G \downarrow \\ BG \end{array}$$

total space EG is (weakly) contractible.

<u>Theorem</u> Let (π_G, EG, BG) be a universal G-bundle, then there is a bijection

$$\Phi : [B, BG] \xrightarrow{\sim} \mathcal{G}(B)$$
$$f \longmapsto f^*\pi_G$$

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\pi"'] & EG \arrow[d, "\pi_G"] \\ B \arrow[r, dashed, "\exists f"'] & BG \end{tikzcd} \qquad \pi \cong f^*\pi_G$$

<u>Proof</u>: Φ is well-defined by the proposition from Tuesday.

Φ is onto. Let $\pi: E \longrightarrow B$ be a principal G-bundle. We must find a map $f: B \longrightarrow BG$ with $f^*\pi_G \simeq \pi$. This is equivalent to finding a bundle morphism $\pi \longrightarrow \pi_G$.

Recall that a morphism

$$\varphi: (\pi, E, B) \longrightarrow (\pi_G, EG, BG)$$

corresponds to a section of the associated bundle

$$E \times_G EG \longrightarrow B$$

with fiber EG. Because EG is weakly contractible, such a section exists by the following lemma.

Lemma: Let $\pi : E \longrightarrow X$ be a fiber bundle with $\pi_i F = 0$ for all $i \geq 0$. If $A \subseteq X$ is a subcomplex, then every section over A extends to a section defined in all of X. In particular, π has a section (take $A = \emptyset$). Moreover, any two sections of π are homotopic.

Proof: Given a section $\sigma_0 : A \longrightarrow E$ of π over A, we extend it to a section $\sigma : X \longrightarrow E$ of π over X by using induction on the dimension of the cells in $X - A$.

So it suffices to assume that

$$X = A \cup_{\varphi} e^n$$

where e^n is an n-cell in $X - A$ with attaching map $\varphi : \partial e^n \longrightarrow A$.

Since e^n is contractible, π is trivial over e^n, so we have a commutative diagram

$$\begin{tikzcd}
 & \pi^{-1}(e^n) \arrow[r, "h", "\cong"'] \arrow[d, shift left, "\pi"] & e^n \times F \arrow[dl, "pr_1"] \arrow[l, bend right, dashed] \\
\partial e^n \arrow[ur, "\sigma_0"] \arrow[r, hook] & e^n \arrow[u, shift left, "\sigma"] \arrow[ur, bend right, dashed, "\sigma"'] &
\end{tikzcd}$$

By composing with h, we can regard σ_0 (for $x \in \partial e^n$) as given by

$$\sigma_0(x) = (x, \tau_0(x)) \in e^n \times F$$

where $\tau_0 : \partial e^n \cong S^{n-1} \longrightarrow F$

Since $\pi_{n-1}(F) = 0$, τ^0 extends to a map $\tau : e^n \longrightarrow F$, which can be used to extend σ_0 over e_n by setting

$$\sigma(x) = (x, \tau(x))$$

After composing with h^{-1} we get the desired extension of σ_0 over e_n.

To show any two sections σ, σ' are homotopic, consider the bundle

$$\pi \times id_I : E \times I \longrightarrow X \times I$$

We can consider σ as a section over $X \times \{0\}$ & σ' as a section over $X \times \{1\}$. Suppose we can construct a section Σ of $\pi \times id_I$ which extends the section over $X \times \{0, 1\}$ defined by (σ, σ'). Then $\Sigma(x, t)$

$$= (\sigma_t(x), t) \quad \text{where} \quad \begin{aligned} \sigma_0 &= \sigma \\ \sigma_1 &= \sigma' \end{aligned}$$

$\Rightarrow$ σ_t will provide the desired homotopy between σ & σ'.

Such a section can be constructed as in the first part of the proof. $\square$.

Φ is injective: Suppose we have

$$f, g : B \longrightarrow BG$$

such that

$$\pi_0 := f^*\pi_G \cong g^*\pi_G =: \pi_1$$

then we must show that $f \simeq g$.

$$\begin{tikzcd}
E_0 = f^*EG \arrow[r, "\hat{f}"] \arrow[d, "\pi_0"'] & EG \arrow[d, "\pi_G"] \\
B \cong B \times \{0\} \arrow[r, "f"'] & BG
\end{tikzcd}$$

Because $\pi_0 \cong \pi_1$

$$\begin{tikzcd}
E_0 \cong E_1 = g^*EG \arrow[r, "\hat{g}"] \arrow[d, "\pi_1"'] & EG \arrow[d, "\pi_G"] \\
B \cong B \times \{1\} \arrow[r, "g"'] & BG
\end{tikzcd}$$

We can combine the two diagrams above

$$\begin{tikzcd}
E_0 \times I \arrow[d, "\pi_0 \times \mathrm{id}"'] & E_0 \times \{0,1\} \arrow[l, hook'] \arrow[r, "\hat{\alpha} = (\hat{f},0) \cup (\hat{g},1)"] \arrow[d, "\pi_0 \times \{0,1\}"] & EG \arrow[d, "\pi_G"] \\
B \times I & B \times \{0,1\} \arrow[l, hook'] \arrow[r, "\alpha = (f,0) \cup (g,1)"'] & BG
\end{tikzcd}$$

It suffices to extend $(\alpha, \hat{\alpha})$ to a bundle morphism $(H, \hat{H}) : \pi_0 \times \mathrm{id} \longrightarrow \pi_G$ as then $H : B \times I \longrightarrow BG$ will give the required homotopy. Using the proposition from Tuesday such a bundle map corresponds to a section of the fiber bundle

$$\omega : (E_0 \times I) \times_G EG \longrightarrow B \times I$$

On the other hand the bundle map $(\alpha, \hat{\alpha})$ corresponds to a section σ_0 of the fiber bundle

$$\omega_0 : (E_0 \times \{0,1\}) \times_G EG \longrightarrow B \times \{0,1\}.$$

There is an inclusion

$$(E_0 \times \{0,1\}) \times_G EG \subseteq (E_0 \times I) \times_G EG$$

so we can regard σ_0 as a section of ω over the subcomplex $B \times \{0,1\}$.

Since EG is contractible, this section can be extended to a section of ω defined on $B \times I$, as desired. $\square$.

<u>Example</u> How many principal G-bundles over S^4 are there?

$$[S^4, BG] \cong \pi_4 BG$$

$$G \longrightarrow EG \xrightarrow{\pi_G} BG$$

Because EG is contractible, the long exact sequence shows $\pi_n BG \cong \pi_{n-1} G$.

Existence of universal bundles

Theorem: Let G be a locally compact ~~topo~~ logical group. Then there exists a universal principal G-bundle $\pi_G : EG \longrightarrow BG$, & the construction is functorial in G, in the sense that $\mu : G \to H$ induces a bundle $(B\mu, E\mu) : \pi_G \to \pi_H$.

Moreover, BG is unique up to homotopy.

Proof: We just show uniqueness up to homotopy. Suppose

$$\pi_G : EG \longrightarrow BG$$

$$\pi_G' : EG' \longrightarrow BG'$$

are universal principal G-bundles.

We can regard π_G as the universal bundle for π_G', so we get a map $f : BG' \longrightarrow BG$ such that $\pi_G' \cong f^* \pi_G$. Likewise we can regard π_G' as the universal bundle for π_G.

$$\begin{tikzcd}
EG \arrow[r] \arrow[d, "\pi_G"'] & EG' \arrow[r] \arrow[d, "\pi_G'"] & EG \arrow[d, "\pi_G"] \\
BG \arrow[r, "g"'] & BG' \arrow[r, "f"'] & BG
\end{tikzcd}$$

$$\pi_G \cong g^* \pi_G' \cong g^* f^* \pi_G = (f \circ g)^* \pi_G$$

$$\cong (\mathrm{id}_{BG})^* \pi_G$$

$$\pi_G \cong (\mathrm{id}_{BG})^* \pi_G$$

By the theorem $f \circ g \simeq id_{BG}$.

Similarly, $g \circ f \simeq id_{BG}$. f & g are homotopy equivalences.

One model is to take

$$EG^n := \underbrace{G * G * \cdots * G}_{n\text{-times}}$$

where $A * B = A \times B \times I / \sim$

$$(a, b_1, 0) \sim (a, b_2, 0)$$
$$(a_1, b, 1) \sim (a_2, b, 1).$$

Milnor showed that EG^n is $(n-1)$-connected & it has a G-action given by right multiplication in each factor of G. The limit $EG = \lim_{n\to\infty} EG^n$ is a weakly contractible G-space, & $\underbrace{BG = EG/G}_{\text{classifying space for the group } G}$.

Rem: G discrete, then $BG \simeq K(G,1)$.

Example $V_n(\mathbb{R}^k)$ = Stiefel manifolds $\{n\text{-frames in } \mathbb{R}^k\}$

$G_n(\mathbb{R}^k) = \{n\text{-dimensional vector subspaces of } \mathbb{R}^k\}$

We saw that there are fibrations

$$O(n) \longrightarrow V_n(\mathbb{R}^k) \longrightarrow G_n(\mathbb{R}^k)$$

Define $V_n(\mathbb{R}^\infty) = \operatorname{colim}_k V_n(\mathbb{R}^k)$

$G_n(\mathbb{R}^\infty) = \operatorname{colim}_k G_n(\mathbb{R}^k)$

$V_n(\mathbb{R}^\infty)$ is contractible, & there is a fibration

$$O(n) \longrightarrow V_n(\mathbb{R}^\infty) \longrightarrow G_n(\mathbb{R}^\infty)$$

This is a universal principle $O(n)$ bundle, because $V_n(\mathbb{R}^\infty)$ is contractible.

In other words $BO(n) \simeq G_n(\mathbb{R}^\infty)$.
In fact $BGL_n(\mathbb{R}) \simeq BO(n)$, so this bundle classifies rank n real vector bundles.

Similarly there is a fiber sequence

$$U(n) \longrightarrow V_n(\mathbb{C}^\infty) \longrightarrow G_n(\mathbb{C}^\infty)$$

$V_n(\mathbb{C}^\infty)$ is contractible, & so $BU(n) \simeq G_n(\mathbb{C}^\infty)$. This classifies rank n complex vector bundles.

<u>Example</u>: Let $G = \mathbb{Z}/2$ & consider the principal $\mathbb{Z}/2$-bundle

$$\mathbb{Z}/2 \longrightarrow S^\infty \longrightarrow \mathbb{RP}^\infty$$

Since S^∞ is contractible, we see that $B\mathbb{Z}/2 \simeq \mathbb{RP}^\infty \simeq K(\mathbb{Z}/2, 1)$.

We see that $\mathbb{R}P^\infty$ classifies rank 1 real vector bundles (real line bundles)

$$\mathrm{Principal}_{\mathbb{Z}/2}(X) = [X, B\mathbb{Z}/2]$$
$$= [X, K(\mathbb{Z}/2, 1)]$$

Recall: $[X, K(G,n)] \cong H^n(X; G)$

$$= H^1(X; \mathbb{Z}/2)$$

for any CW-complex X. Now let π be a real line bundle on X with a classifying map $f_\pi : X \longrightarrow \mathbb{R}P^\infty$.

Recall $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) \cong \mathbb{F}_2[\omega]$ with ω a generator of $H^1(\mathbb{R}P^\infty; \mathbb{Z}/2)$,

By pullback along f_π we get a well-defined degree 1 class

$$\omega_1(\pi) := f_\pi^*(\omega)$$

the first Stiefel-Whitney class.

Under the bijection

$$\mathrm{Principal}_{\mathbb{Z}/2}(X) \cong H^1(X;\mathbb{Z}/2)$$
$$\pi \longmapsto w_1(\pi).$$

i.e. real line bundles are completely classified by their first Stiefel-Whitney class.

Example (complex line bundles)

There is a principal S^1-bundle

$$S^1 \longrightarrow S^\infty \longrightarrow \mathbb{C}P^\infty$$

shows that $BS^1 \simeq \mathbb{C}P^\infty = K(\mathbb{Z},2)$.

$$\begin{aligned}\mathrm{Principal}_{S^1}(X) &= [X, BS^1] \\ &= [X, K(\mathbb{Z},2)] \\ &\cong H^2(X;\mathbb{Z})\end{aligned}$$

We have $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[c]$, $|c| = 2$

Any principal S^1-bundle over X gives rise to a map $f_\pi : X \longrightarrow BS^1$
$= \mathbb{C}P^\infty$

$$c_1(\pi) := f_\pi^*(c) \in H^2(X; \mathbb{Z})$$

the first Chern class. Under the isomorphism

$$\mathrm{Principal}_{S^1}(X) \xrightarrow{\sim} H^2(X; \mathbb{Z})$$
$$\pi \longmapsto c_1(\pi)$$

$\Rightarrow$ Complex line bundles are completely classified by their first Chern class.

Post-lecture remark

$G(-)$ defines a functor $h\mathrm{Top} \longrightarrow \mathrm{Set}$.

A