Week 10

Exercise: Suppose I have a morphism of fiber bundles

$$\begin{tikzcd} E' \arrow[r, "\varphi'"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ B' \arrow[r, "\varphi"] & B \end{tikzcd}$$

Then, $\pi' \cong \varphi^* \pi$.

Associated bundle

Let (π, P, B) be a principal G-bundle, & let $e: G \to \mathrm{Aut}(F)$ a left action on a space F. Then the product $P \times F$ has a canonical right action

$$(p, f) \cdot g = (p \cdot g, \underbrace{e(g^{-1}) \cdot f}_{g^{-1} \cdot f})$$

Prop: The quotient of $P \times F$ by G defines a fiber bundle over B with fiber F by

$$P \times_G F := (P \times F)/G \xrightarrow{\pi_F} B$$
$$[p, f] \longmapsto \pi(p)$$

This is called the associated fiber bundle. If P is trivial as a G-bundle, then $P \times_G F$ is trivial for any F.

Proof: We first check that π_F is well-defined. Suppose that $[p', f']$ is another representative in this equivalence class. Then $p' = p \cdot g$ for some $g \in G$
$$f' = g^{-1} \cdot f$$

$$\begin{aligned}
\pi_F([p', f']) &= \pi_F([p \cdot g, g^{-1} \cdot f]) \\
&= \pi(p \cdot g) \\
&= \pi(p) \\
&= \pi_F([p, f])
\end{aligned}$$

We claim that the fibers of $P \times_G F$ are homeomorphic to F. To see this, fix a point $b \in B$, & choose a point $p_0 \in \pi^{-1}(b)$ in the fiber of P over B. We define a cts map

$$\begin{aligned} F &\longrightarrow \pi_F^{-1}(b) \\ f &\longmapsto [p_0, f] \end{aligned}$$

This has a cts inverse given by

$$\begin{aligned} \pi_P^{-1}(b) &\longrightarrow F \\ [p, f] &\longmapsto \tau(p_0, p) \cdot f \end{aligned}$$

where $\tau(p_0, p) \in G$ is the unique element such that $p_0 \cdot \tau(p_0, p) = p$.

$$[p_0, f] \longmapsto [p_0, \underbrace{\tau(p_0, p_0)}_{e} \cdot f]$$

The map $\pi^{-1}(B) \times F \ni (p, f) \longmapsto \tau(p_0, p) \cdot f \in F$

is invariant with respect to the

G-action

$$(p, f) \cdot g = (p \cdot g, \ g^{-1} \cdot f)$$

$$\longmapsto \tau(p_0, p \cdot g)\, g^{-1} \cdot f$$

$$= \tau(p_0, p) \cdot g \cdot g^{-1} \cdot f$$

$$= \tau(p_0, p) \cdot f$$

& hence this descends to the quotient

$$\pi^{-1}(B) \times F / G = \pi_F^{-1}(B).$$

To finish the proof it suffices to prove the final claim: triviality of P implies triviality of $P \times_G F$.

Hence we can assume $P = B \times G$.

Then $(P \times F)/G = (B \times G \times F)/G$

This is isomorphic to $B \times F$

$$([b,g,f]) \longmapsto (b, g\cdot f)$$

with inverse

$$(b,f) \longmapsto [(b,e,f)]. \qquad \square.$$

Remark (structure group of a bundle).

Suppose B is covered by a collection sets $\{U_\alpha\}$ with local trivializations

$$\varphi_\alpha : U_\alpha \times F \xrightarrow{\cong} p^{-1}(U_\alpha)$$

Let's compare the local trivializations on the intersection $U_\alpha \cap U_\beta$.

$$\begin{tikzcd} U_\alpha \cap U_\beta \times F \arrow[r, "\cong", "\varphi_\beta"'] \arrow[dr, dashed] & p^{-1}(U_\alpha \cap U_\beta) \arrow[d, "\cong"', "\varphi_\alpha^{-1}"] \\ & U_\alpha \cap U_\beta \times F \end{tikzcd}$$

For every $x \in U_\alpha \cap U_\beta$, we obtain a homeomorphism of the fiber F,

a map
$$\varphi_{\alpha,\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Aut}(F)$$
these are called <u>clutching functions</u>.

If the bundle is an n-dimensional vector-bundle, then the clutching functions take the form
$$\varphi_{\alpha,\beta} : U_\alpha \cap U_\beta \longrightarrow GL(n, \mathbb{F})$$

If $p : E \longrightarrow B$ is a principal G-bundle, then the clutching functions take values in G:
$$\varphi_{\alpha,\beta} : U_\alpha \cap U_\beta \longrightarrow G \subset \mathrm{Aut}(F).$$

In general, a fiber bundle with structure group G is one whose clutching functions take values in G.

Prop: Given any fiber bundle $\pi: E \longrightarrow B$ with fiber F & structure group $\mathrm{Aut}(F)$, there exists a principal $G := \mathrm{Aut}(F)$-bundle P such that $E = P \times_G F$.

Proof: For $b \in B$, define

$$\underbrace{P_b = \{ G\text{-isomorphisms } \varphi : F \longrightarrow \pi^{-1}(b) \}}_{\text{set of frames}}$$

This has an action of G on the right

$$\varphi \cdot g : F \longrightarrow \pi^{-1}(b)$$

is also a G-isomorphism

This is a free & transitive action:

any two $\varphi, \varphi' : F \longrightarrow \pi^{-1}(b)$

are related by $g = \varphi'^{-1} \circ \varphi \in G = \mathrm{Aut}(P)$.

Define

$$P = \bigcup_{b \in B} P_b$$

& define

$$\pi_P : P \longrightarrow B$$
$$p \in P_b \longmapsto b$$

Suppose $E = B \times F$, then $\pi^{-1}(b) = \{b\} \times F$

& canonically, $P_b \cong G$. Thus in this case

$$P = \bigcup_{b \in B} P_b = \bigcup_{b \in B} \{b\} \times G$$
$$= B \times G$$

In the general case we do the same construction over local trivializations.

To see that $P \times_G F \cong E$, note that points in $P \times_G F$ are equivalence classes $[b, \varphi, f]$ where $b \in B$, $\varphi : F \longrightarrow \pi^{-1}(b)$ is a G-isomorphism & $f \in F$.

Now consider

$$P \times_G F \ni [b, \varphi, f] \longmapsto \varphi(f) \in E.$$

This is well-defined

$$[b, \varphi \cdot g, g^{-1} \cdot f] \longmapsto \varphi(g \cdot g^{-1} f)$$
$$= \varphi(f)$$

Because φ is an isomorphism, this gives the bundle isomorphism. $\square$.

Notation: F is a space with a left G-action, a cts map $f : P \longrightarrow F$ is G-equivariant if $f(p \cdot g) = g^{-1} \cdot f(p)$. We denote the set of such maps by $\mathrm{Map}_G(P, F)$.

Prop: Let (π, P, B) be a principal G-bundle, F a space with a left G-action, $\triangle$ $E = P \times_G F$ the associated bundle. Then there is a bijection

$$\Gamma(U, E) \rightleftarrows \mathrm{Map}_G(\pi^{-1}(U), F).$$

Proof: Given $\tilde{s} \in \mathrm{Map}_G(\pi^{-1}(U), F)$ define

$$s(b) := [p, \tilde{s}(p)]$$

This is well-defined

$$[p \cdot g, \tilde{s}(p \cdot g)] = [p \cdot g, g^{-1} \tilde{s}(p)]$$
$$= [p, \tilde{s}(p)].$$

Conversely, let $s : U \longrightarrow E$ be a local section. Suppose $s(\pi(p)) = [p, f]$

Define

$$\tilde{s} : \pi^{-1}(U) \longrightarrow F$$
$$p \longmapsto f$$

To see this is G-equivariant

$$\tilde{s}(p \cdot g) = g^{-1} \cdot \tilde{s}(p)$$

because $s(\pi(p \cdot g)) = s(\pi(p))$

$$= [p, f]$$

$$= [p \cdot g, g^{-1} \cdot f]$$

From the defintions s & $\hat{s}$ are inverse bijections. Also, s cts $\Leftrightarrow$ $\hat{s}$ cts $\square$.

<u>Prop</u>: Fix a group G. Let (π, P, B) & (π', Q, B') be principal G-bundles over B & B', Then there is a bijection between bundle morphisms $\varphi: (\pi, P, B) \longrightarrow (\pi', Q, B')$ & sections of the associated

↗ Q is a left G-space with action $g \cdot q = q \cdot g^{-1}$.

bundle $P \times_G Q$

<u>Proof</u>: A morphism as above is specified by a G-map $\phi : P \longrightarrow Q$ i.e. a element $\mathrm{Map}_G(P, Q)$.

$$\mathrm{Map}_G(P, Q) = \mathrm{Map}_G(\pi^{-1}(B), Q)$$

[Previous prop $\Rightarrow$] $= \Gamma(B, P \times_G Q)$.

Homotopy classification of principal bundles

<u>Prop</u>: If (π, P, B') a principal G-bundle, & $f_0 \simeq f_1 : B \longrightarrow B'$, then $f_0^*(\pi)$ & $f_1^*(\pi)$ are isomorphic as bundles over B.

<u>Rem</u>: Use the associated bundle

constructions, can prove it holds for any fiber bundle.

Exercise: Fiber bundles over a contractible base space are trivial.

Proof of proposition:

Let $f_t : B \times I \longrightarrow B$ be a homotopy between f_0 & f_1. Now consider the following diagram:

$$
\begin{tikzcd}
f_0^* P \arrow[rr] \arrow[dd, "f_0^* \pi"'] & & f_t^* P \arrow[r] \arrow[dd, "f_t^* \pi"'] & P \arrow[dd, "\pi"] \\
& f_1^* P \arrow[ur] \arrow[dd, "f_1^* \pi"'] & & \\
B \times \{0\} \arrow[rr, hook, "i_0"'] & & B \times I \arrow[r, "f_t"'] & B \\
& B \times \{1\} \arrow[ur, hook, "i_1"'] & &
\end{tikzcd}
$$

<u>Claim</u>: For a principal G-bundle $(\pi, Q, B\times I)$ we have $i_0^*\pi \simeq i_1^*\pi$

$i_0 : B\times\{0\} \longrightarrow B\times I$
$i_1 : B\times\{1\} \longrightarrow B\times I$

This will complete the proof as then

$$f_0^*\pi \simeq i_0^* f_t^*\pi \simeq i_1^* f_t^*\pi \simeq f_1^*\pi$$

<u>Proof of the claim</u>:

$$\begin{tikzcd}
i_0^*\pi \arrow[rr] \arrow[dd, "\pi_0"'] & & Q \arrow[dd, "\pi"] \\
& i_1^*\pi \arrow[ur] \arrow[dd, "\pi_1"'] & \\
B\times\{0\} \arrow[rr, hook, "i_0"'] & & B\times I \\
& B\times\{1\} \arrow[ur, hook, "i_1"'] &
\end{tikzcd}$$

We claim in fact that

$$\pi \cong \pi_0 \times id_I$$

If this holds, then

$$Q|_{B\times\{1\}} = i_1^*\pi$$
$$= (Q\times I_0)|_{B\times I} \simeq i_0^*\pi$$

Since we work with principal G-bundles morphisms over a fixed base are isomorphisms. Therefore it suffices to produce a morphism

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & Q_0 \times I \arrow[d] \\ B \times I \arrow[r, "id"] & B \times I \end{tikzcd}$$

By the previous proposition, this corresponds to a section of the bundle

$$\omega : Q \times_G (Q_0 \times I) \longrightarrow B \times I.$$

Now, by definition, $Q \times_G (Q_0 \times I)$ has a section s over $B \times \{0\}$. Now consider the following diagram

$Q|_{B \times \{0\}}$ & $Q_0 \times I|_{B \times \{0\}}$ are both just Q_0.

$$\begin{tikzcd}
B \times \{0\} \arrow[r, "s"] \arrow[d, "i_0"'] & Q \times_G (G_0 \times I) \arrow[d, "\omega"] \\
B \times I \arrow[r, "\text{identity}"'] \arrow[ur, dashed, "\tilde{s}"] & B \times I
\end{tikzcd}$$

We want to find a lift $\tilde{s}$ in this diagram. The lift $\tilde{s}$ exists by the homotopy lifting property, because ω is a fibration. $\square$.

Rem: $\mathcal{G}(B)$ = set of principal G-bundles over B. This is functorial with respect to cts maps $f: A \longrightarrow B$

$$P \longmapsto f^*(P)$$

The previous proposition shows that $\mathcal{G}$ descends to the homotopy category.

$$\mathcal{G} : \mathrm{hTop} \longrightarrow \left\{ \begin{array}{c} \text{principal } G\text{-bundles} \\ \text{up to isomorphism} \end{array} \right\}.$$