$$\Omega S^n \longrightarrow PS^n \longrightarrow S^n$$

$$E_2^{p,q} \cong H^p(S^n; H^q(\Omega S^n)) \Rightarrow H^{p+q}(PS^n)$$

$$\begin{tikzcd}
3n-3 & \mathbb{Z}\{a_3\} \arrow[rrd, "\cong"] & & \mathbb{Z}\{a_3 x\} \\
2n-2 & \mathbb{Z}\{a_2\} \arrow[rrd, "\cong"] & & \mathbb{Z}\{a_2 x\} \\
n-1 & \mathbb{Z}\{a_1\} \arrow[rrd, "\cong", "d_n"'] & & \mathbb{Z}\{a_1 x\} \\
0 & \mathbb{Z}\{1\} & & \mathbb{Z}\{x\} \\
 & 0 & & n
\end{tikzcd}$$

Rem: Cup product is graded-commutative, so

$$a \cdot b = (-1)^{|a|\cdot|b|}\, b \cdot a$$

By replacing some a_k's with their negatives we can assume

$$d_n(a_1) = x$$
$$d_n(a_k) = a_{k-1} x \quad \text{for } k > 1.$$

We also note $a_k x = x a_k$, because $|a_k|\,|x|$ = even

n odd

$$d_n(a_1^2) = d_n(a_1)\cdot a_1 + \overbrace{(-1)^{n-1}}^{=1} a_1 \cdot d_n(a_1)$$
$$= x \cdot a_1 + a_1 \cdot x = 2 \cdot a_1 \cdot x$$

$$d_n(a_2) = a_1 x$$
$$\Rightarrow a_1^2 = 2a_2, \quad \text{because } d_n \text{ is an iso.}$$

$$d_n(a_1^k) = d_n(a_1 \cdot a_1^{k-1}) = d_n(a_1)\cdot a_1^{k-1} + (-1)^{|a_1|} a_1 \cdot d_n(a_1^{k-1})$$
$$= x \cdot a_1^{k-1} + a_1 \cdot d_n(a_1^{k-1})$$

$$\bigstar \quad a_k = a_1^k / k!$$

$$a_m \cdot a_n = a_1^m / m! \cdot a_1^n / n!$$

$$= \frac{a_1^{m+n}}{m!\, n!}$$

$$= x \cdot a_1^{k-1} + a_1 \left[d_n(a_1) a_1^{k-2} + a_1 \cdot d_n(a_1^{k-2}) \right]$$

$$= x \cdot a_1^{k-1} + a_1^{k-1} x + a_1 d_n(a_1^{k-2})$$

$$= 2x a_1^{k-1} + a_1 d_n(a_1^{k-2})$$

$$\vdots$$

$$= k a_1^{k-1} x$$

By induction $\quad a_1^k = k!\, a_k$

$$\Rightarrow H^*(\Omega S^n; \mathbb{Z}) \cong \Gamma_{\mathbb{Z}}[a], \quad |a| = n-1, \qquad a \leftrightarrow a_1 \quad \bigstar$$

n even $\quad |a_1|$ is odd, $\quad |a_1| \cdot |a_1| = (-1) \cdot |a_1| \cdot |a_1|$

$$\Rightarrow a_1^2 = 0.$$

$\boxed{a_1 a_2}$

$$d_n(a_1 a_2) = d_n(a_1) \cdot a_2 + (-1)^{|a_1|} a_1 \cdot d_n(a_2)$$

$$= x \cdot a_2 - a_1^2 x$$

$$= x \cdot a_2$$

$$= d_n(a_3)$$

$$\therefore a_1 a_2 = a_3$$

Now claim inductively

$$a_1 a_{2k} = a_{2k+1} \qquad [\Delta \text{ hence } \quad a_1 a_{2k+1} = a_1 [a_1 a_{2k}] = 0$$

Indeed,

$$d_n(a_1 a_{2k}) = x \cdot a_{2k} - \overbrace{a_1 \cdot a_{2k-1}}^{0} \cdot x$$

$$= x \cdot a_{2k} = d_n(a_{2k+1}) \Rightarrow a_1 a_{2k} = a_{2k+1}$$

$$a_1 a_{2k+1} = 0$$

The second claim is $a_2^k = k!\, a_{2k}$.

We have $d_n(a_2^k) = d_n(a_2^1 \cdot a_2^{k-1}) = d_n(a_2)\, a_2^{k-1} + (-1)^{|a_2|} a_2 \cdot d_n(a_2^{k-1})$

$= a_1 \cdot x \cdot a_2^{k-1} + a_2\, d_n(a_2^{k-1})$

$= k \cdot a_1 \cdot x \cdot a_2^{k-1}$, inductively

By induction $a_2^{k-1} = (k-1)!\, a_{2k-2}$ & so

$$d_n(a_2^k) = k \cdot a_1 \cdot x \cdot (k-1)!\, a_{2k-2}$$
$$= k! \cdot a_1 \cdot x \cdot a_{2k-2}$$
$$= k!\, a_{2k-1} \cdot x \qquad [\text{using } a_1 \cdot a_{2k-2} = a_{2k-1}]$$
$$= k!\, d_n(a_{2k})$$
$$\Rightarrow a_2^k = k!\, a_{2k}, \text{ as claimed.}$$

$$H^*(\Omega S^n; \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[a] \otimes \Gamma_{\mathbb{Z}}[b]$$

$|a| = n-1 \qquad |b| = 2n-2$

$a \longleftrightarrow a_1 \qquad b \longleftrightarrow a_2$