MA3408 Week 11

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Question 1.

If $f\colon Y \to X$ is a continuous map and $\pi\colon E \to X$ is a $U(n)$-bundle, then $c_i(f^*\pi) = f^*c_i(\pi)$ for any i.

Remark: Note that f^* has two different meanings here, once as a vector bundle pullback, and once as the pullback of a cohomology class.

Proof. The commutative diagram

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d, "f^*\pi"'] & E \arrow[r] \arrow[d, "\pi"'] & EU(n) \arrow[d, "\pi_{U(n)}"] \\ Y \arrow[r, "f"'] & X \arrow[r, "f_\pi"'] & BU(n) \end{tikzcd}$$

shows that $f_\pi \circ f$ is the classifying map from the bundle $f^*\pi$ over Y. Therefore,

$$c_i(f^*\pi) = (f_\pi \circ f)^*(c_i) = f^*(f_\pi^* c_i) = f^*(c_i(\pi)),$$

as claimed. $\square$

Question 2.

Show that $B(G \times H) \simeq BG \times BH$ (whenever this makes sense).

Proof. This follows from the homotopy uniqueness of classifying spaces, along with the fact that $EG \times EH$ is a weakly contractible space with a $G \times H$ action for which $(EG \times EH)/G \times H = BG \times BH$. $\square$

Question 3.

Compute the first Chern class of the Hopf bundle $S^1 \to S^3 \to S^2$.

Proof. One way to do this is to use the lemma we proved in class; for a principal S^1-bundle, we have that $c_1(\pi) = d_2(a)$ where $a \in H^1(S^1, \mathbb{Z})$ is a generator. We have already studied the associated Serre spectral sequence, and seen that $d_2(a)\colon \mathbb{Z} \to \mathbb{Z}$ is an isomorphism, and so is given by multiplication by ± 1. Under our conventions, it is equal to one, and so $c_1(\pi) = 1$.[1] $\square$

[1] This relates to the 4th axiom; we can choose $c_1(\gamma_1^1)$ to be isomorphic to the canonical generator of $H^*(\mathbb{C}P^\infty)$ or its negative.