MA3408 Week 10

Drew Heard (drew.k.heard@ntnu.no)

March 23, 2021

Question 1.

Show that a fibre bundle over a contractible base space is trivial.

Proof. Let $\pi\colon E \to B$ be a fiber bundle with B contractible. Let $f\colon B \to B$ be a map taking the whole space B to a point $b \in B$. Then f and id_B are homotopic by assumption. Therefore, the bundles $f^*\pi$ and $\mathrm{id}_B^*\pi$ are isomorphic (by the proposition we did in class). Obviously $\mathrm{id}_B^*\pi \cong \pi$, while $f^*\pi$ is a trivial bundle. $\square$

Question 2.

If

$$\begin{array}{ccc} E' & \xrightarrow{\tilde{f}} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{f} & B \end{array}$$

is a morphism of principal G-bundles, then $\pi' \cong f^*\pi$ as bundles over B.

Remark: More generally, it should hold whenever the morphism is a fiberwise isomorphism (I think!).

Proof. It is enough to construct a G-map $E' \to f^*E$, as this gives a principal G-bundle morphism over B, which is an isomorphism (principal G-bundle morphisms over a fixed base are isomorphisms). We define such a map $h\colon E' \to f^*E$ sending $e' \mapsto (\pi'(e'), \tilde{f}(e')) \in B' \times E$. We need to check that this is well-defined, i.e. $f \circ \pi'(e') \cong \pi \circ \tilde{f}(e')$, but this hold because the diagram is assumed to commute. Finally this is a G-map by definition of the actions. $\square$

Question 3.

Show that there are exactly two principal $SO(3)$-bundles over S^2.

Proof. By the classification theorem we have that these are classified by $\pi_2(BSO(3)) \cong \pi_1(SO(3))$. Then we note that $SO(3)$ is homeomorphic to $\mathbb{R}P^3$, to deduce that $\pi_2(BSO(3)) \cong \mathbb{Z}/2$ so that (up to isomorphism) there are two such bundles. $\square$