Introduction to Lie theory

Material for week 9

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What we learned so far

Constructed the tangent manifold TM to a smooth manifold M.

For a smooth map $f\colon M \to N$ we defined the tangent map Tf which locally (in charts!) looks like

$$(\psi \circ f \circ \varphi^{-1}, \mathbf{d}(\psi \circ f \circ \varphi^{-1}))$$

Local on a manifold = Use charts and work in $\mathbb{R}^d$!

Have now: manifolds, tangent maps and some tools (submersions!). Still need differential equations on manifolds.

2.3. Vector fields, their Lie bracket and differential equations

2.3.5 Integral curves and flows

For a C^1-curve $c\colon (a,b) \to M$ we can define $\dot{c}(t) := T_t c(1) \in T_{c(t)}M$, where we use $T(a,b) = (a,b) \times \mathbb{R}$.

Let $X \in \mathcal{V}(M)$, we say a C^1-curve $c\colon (a,b) \to M$ is an *integral curve* for X if for every $t \in (a,b)$ the curve satisfies $\dot{c}(t) = X(c(t))$.

For every $m \in M$ there exists an integral curve c_m of X on some open interval $J_m :=]-\varepsilon, \varepsilon[$ such that $c_m(0) = m$.

Moreover, the *flow*

$$\mathrm{Fl}^X\colon \bigcup_{m \in M} \{m\} \times J_m \to M, (m,t) \mapsto c_m(t),$$

defines a continuous map on some open subset of $M \times \mathbb{R}$. If Fl^X is defined on all of $M \times \mathbb{R}$, then we call X *complete*.

Theorems on flows of vector fields

2.3.6 Theorem

Let M be a manifold. Any vector field $X \in \mathcal{V}(M)$ admits a unique smooth flow $\mathrm{Fl}^X \colon D_X \to M$, where D_X is an open subset of $M \times \mathbb{R}$.

Moreover, if X smoothly depends on parameters, then so does the flow Fl^X.

2.3.15 Lemma

If $X, Y \in \mathcal{V}(M)$ are complete vector fields then

$$\mathrm{Fl}^X_t \circ \mathrm{Fl}^Y_s = \mathrm{Fl}^Y_s \circ \mathrm{Fl}^X_t, \quad \forall s, t \in \mathbb{R} \text{ if and only if } [X, Y] = 0.$$

Aim: Construct a Lie bracket $[\cdot, \cdot]$ on $\mathcal{V}(M)$ turning it into a Lie algebra.

Lie derivative and Lie bracket

For $X \in \mathcal{V}(M)$ and $f \in C^\infty(M.\mathbb{R}^n)$ define the *Lie derivative*

$$\mathcal{L}_X(f)\colon M \to \mathbb{R}^n, \mathcal{L}_X(f)(p) = \mathrm{pr}_2(Tf(X(p))).$$

For $U \subseteq \mathbb{R}^n$ open, $X, Y \in \mathcal{V}(U)$ with principal parts $\tilde{X}, \tilde{Y}$ define

$$[\tilde{X}, \tilde{Y}] = \mathcal{L}_X(\tilde{Y}) - \mathcal{L}_Y(\tilde{X})$$

Since $T\tilde{Y} = (\tilde{Y}, \mathbf{d}\tilde{Y})$ we have

$$[\tilde{X}, \tilde{Y}](p) = \mathbf{d}\tilde{Y}(p)(\tilde{X}(p)) - \mathbf{d}\tilde{X}(p)(\tilde{Y}(p))$$

So clearly $[\tilde{X}, \tilde{X}] = 0$, but to establish the other properties of the Lie bracket we need more techniques.

Related vector field Lemma

2.3.10 Lemma

Let $U \subseteq \mathbb{R}^d$, $V \subseteq \mathbb{R}^n$ be open and $f \in C^\infty(U, V)$, $X_1, X_2 \in C^\infty(U, \mathbb{R}^d)$ and $Y_1, Y_2 \in C^\infty(V, \mathbb{R}^n)$. Assume that X_i is f-related to Y_i for $i = 1, 2$, then $[X_1, X_2]$ is f-related to $[Y_1, Y_2]$.

Reminder: Relatedness was $Tf \circ X = Y \circ f$ for maps ($X = (\mathrm{id}, \tilde{X})$ and $Y = (\mathrm{id}, \tilde{Y})$ means that

$$\underbrace{\mathbf{d}f(p)(\tilde{X}(p))}_{=\mathcal{L}_X(f)(p)} = \tilde{Y}(f(p)).$$

An annoying formula (Exercise A.2.1)

If f is C^2 and h is C^1 we can define a C^1-function: $\phi(x) = \mathbf{d}f(x)(h(x))$ then

$$\mathbf{d}\phi(x)(y) = \mathbf{d}^2 f(x)(h(x), y) + \mathbf{d}f(x)(\mathbf{d}h(x)(y)) \qquad (A.3)$$

In other words:

$$\mathbf{d}f(p)(\mathbf{d}h(p)(y)) = \mathbf{d}(\mathbf{d}f(p)(h(p)))(y) - \mathbf{d}^2 f(p)(h(x), y)$$

Apply this to the right hand side of

$$\mathbf{d}f(p)([X_1, X_2](p)) = \mathbf{d}f(p)(\mathbf{d}X_2(p)(X_1(p))) - \mathbf{d}f(p)(\mathbf{d}X_1(p)(X_2(p)))$$

Algebras and their Lie bracket

Definition: $\mathbb{K}$-algebra

A $\mathbb{K}$-vector space A is called an associative algebra if there is a bilinear map $\bullet\colon A \times A \to A$ such that

$$(a \bullet b) \bullet c = a \bullet (b \bullet c), \quad \forall a, b, c \in A$$

Some examples

$(\mathbb{K}, \cdot)$ usual multiplication, $(M_n(\mathbb{K}), \cdot)$ matrix multiplication.
$(\mathrm{Lin}(V, V), \circ)$ function composition of linear functions.

$(C^\infty(M, \mathbb{R}), \cdot)$ pointwise multiplication $(f \cdot g)(m) := f(m) \cdot g(m)$

Every associative algebra $(A, \bullet)$ is a Lie algebra, since

$$[a, b] := a \bullet b - b \bullet a$$

is a Lie bracket.

2.3.11 Derivations of an algebra

Derivation of an algebra

If $(A, \cdot)$ is an algebra, $D \in \mathrm{Lin}(A, A)$ is called *derivation* if

$$D(a \cdot b) = a \cdot D(b) + D(a) \cdot b, \qquad a, b \in A.$$

We write $\mathrm{der}(A)$ for the set of derivations of an algebra.

Lemma (Exercise in algebra)

If $(A, \cdot)$ is an associative algebra, $\mathrm{der}(A)$ is a Lie subalgebra of $(\mathrm{Lin}(A, A), [\cdot, \cdot])$, where $[F, G] = F \circ G - G \circ F$.

Vector fields as derivations

Taking $(C^\infty(M, \mathbb{R}), \cdot)$ as an algebra with the pointwise multiplication, we have

$$\mathcal{L}_X(f \cdot g) = f \cdot \mathcal{L}_X(g) + \mathcal{L}_X(f) \cdot g, \quad X \in \mathcal{V}(M)$$

3. Lie groups beyond matrix groups

3.0.1 Definition

A *Lie group* G is a manifold G endowed with a group structure such that the multiplication map $m_G\colon G \times G \to G$ and the inversion map $\iota\colon G \to G$ are smooth.A morphism of Lie groups is a smooth group homomorphism.

3.0.2 Example

For every $d \in \mathbb{N}_0$ every vector space $\mathbb{R}^d$ is a Lie group. Every linear map $f\colon \mathbb{R}^d \to /R^n$ is Lie group morphism.

3.0.3 Example (Consequence of Proposition 1.1.2)

For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ the group $\mathrm{GL}_n(\mathbb{K})$ is a Lie group.

Standard Notation

Let G be a Lie group, we shall write

- $\mathbf{1}_G$ for the unit element (or shorter $\mathbf{1}$),
- m_G for multiplication, ι_G for inversion,
- for $g \in G$ we let $\lambda_g \colon G \to G, h \mapsto gh$ and $\rho_g \colon G \to G, h \mapsto hg$ the *left (right) translation*.

(Observe that $\lambda_g(\rho_h(x)) = gxh = \rho_h(\lambda_g(x))$.)