Introduction to Lie theory

Material for week 6

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23. September 2024

What we learned so far...

Let G be a linear Lie group. Then

$$\mathbf{L}(G) = \{X \in M_n(\mathbb{K})\colon \exp(tX) \in G, \forall t \in \mathbb{R}\}$$

The restriction of the matrix exponential

$$\exp_G\colon \mathbf{L}(G) \to G$$

is called *Lie group exponential* of G.

1.7.3 Proposition

Let G be a linear Lie group and $\exp_G$ its Lie group exponential. There exists an open 0-neighborhood $V_1 \subseteq M_n(\mathbb{K})$ and an open I_n-neighborhood V_2 such that $\exp_G$ restricts to a homeomorphism $V_1 \cap \mathbf{L}(G) \to V_2 \cap G$.

1.8 Interplay of linear Lie groups and Lie algebras

The Lie functor

Define for a Lie group morphism φ a Lie algebra morphism $\mathbf{L}(\varphi)$
Aim: Construct the Lie functor

$$\begin{aligned} \mathbf{L} \colon \textbf{lin. Lie grp} &\to \textbf{Lie algebras} \\ G \text{ linear Lie group} &\mapsto \mathbf{L}(G) \\ \varphi \text{ cont. group morphism} &\mapsto \mathbf{L}(\varphi) \text{ Lie alg. morphism} \end{aligned}$$

That $\mathbf{L}$ is a (covariant) functor means that whenever composition is defined we have

$$\mathbf{L}(\varphi \circ \psi) = \mathbf{L}(\varphi) \circ \mathbf{L}(\psi), \text{ and}$$

$$\mathbf{L}(\mathrm{id}_G) = \mathrm{id}_{\mathbf{L}(G)}$$

Proposition 1.8.1

Let G_1, G_2 be linear Lie groups and $\varphi\colon G_1 \to G_2$ a continuous group homomorphism. Then the derivative

$$\mathbf{L}(\varphi)(x) := \left.\frac{d}{dt}\right|_{t=0} \varphi(\exp_{G_1}(tx))$$

exists for each $x \in \mathbf{L}(G_1)$ and defines a morphism of Lie algebras $\mathbf{L}(\varphi)\colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ which satisfies

$$\exp_{G_2} \circ \mathbf{L}(\varphi) = \varphi \circ \exp_{G_1} \qquad (1.10)$$

Furthermore, $\mathbf{L}(\varphi)$ is the uniquely determined linear map satisfying (1.10).

1.8.2 Remark: Naturality of the exponential

Expressing (1.10) as a commutative diagram we get

$$
\begin{array}{ccc}
\mathbf{L}(G_1) & \xrightarrow{\mathbf{L}(\varphi)} & \mathbf{L}(G_2) \\
\Big\downarrow{\scriptstyle \exp_{G_1}} & & \Big\downarrow{\scriptstyle \exp_{G_2}} \\
G_1 & \xrightarrow{\quad\varphi\quad} & G_2
\end{array}
$$

This property is called "naturality of the Lie group exponential"

Functorial properties

1.8.3 Lemma (Proof left as an exercise)

If G_1, G_2, G_3 are linear Lie groups and $\varphi_1\colon G_1 \to G_2$ and $\varphi_2\colon G_2 \to G_3$ are continous group homomorphisms, then

$$\mathbf{L}(\mathrm{id}_G) = \mathrm{id}_{\mathbf{L}(G)}, \qquad \mathbf{L}(\varphi_2 \circ \varphi_1) = \mathbf{L}(\varphi_2) \circ \mathbf{L}(\varphi_1).$$

This is the functor property of $\mathbf{L}$.

1.8.4 Corollary

If $\varphi\colon G_1 \to G_2$ is an isomorphism of linear Lie groups, then $\mathbf{L}(\varphi)$ is an isomorphism of Lie algebras.

1.9 The Baker-Campbell-Dynkin-Hausdorff Formula

A formula for the local multiplication

We used a "local multiplication" in some of the proofs:

$$x * y = \log(\exp_G(x) \exp_G(y)), \qquad x, y \in \mathbf{L}(G)$$

The Baker-Campbell-Dynkin-Hausdorff (BCDH) Formula allows us to express this as a series of nested Lie brackets:

$$x * y = x + y + \frac{1}{2}[x, y] + \frac{1}{12}[x, [x, y]] + \frac{1}{12}[y, [y, x]] + \cdots .$$

Note: If $[x, y] = 0$, the BCDH formula reads $x * y = x + y$ applying the (Lie group) exponential to both sides we recover

$$\exp_G(x + y) = \exp_G(x) \exp_G(y).$$

Some notation

To write the nested Lie brackets it is customary to write

$$\mathrm{ad}(x)(y) := [x, y]$$

Then $\mathrm{ad}(x)^2(y) = [x, [x, y]]$ and $\mathrm{ad}(x) \circ \mathrm{ad}(y)(z) = [x, , [y, z]]$, etc.

To compute the BCDH-formula one needs an inverse of $\mathbf{d}\exp$. It turns out that the complex series

$$\Psi(z) = \frac{z \log(z)}{z-1} := z \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}(z-1)^k \quad \text{for } |z-1| < 1, z \in \mathbb{X}$$

appears in the computation:

1.9.3 A technical Lemma

Let $X \in M_n(\mathbb{K})$. Then

$$\mathbf{d}\exp(X) = \lambda_{\exp(X)} \circ \int_0^1 e^{-s\mathrm{ad}(X)}\mathrm{d}s = \lambda_{\exp(X)} \circ \sum_{n=0}^{\infty}(-1)^n \frac{\mathrm{ad}(X)^m}{(m+1)!}$$

$$= \Psi(\mathrm{ad}(X))$$

1.9.4 Theorem (BCDH-formula)

For $X, Y \in M_n(\mathbb{K})$ with $\|X\|, \|Y\|$ sufficiently small the integral formulation of the BCDH-formula holds

$$X * Y = \log\left(e^X e^Y\right) = X + \int_0^1 \Psi(e^{\mathrm{ad}(X)} e^{\mathrm{ad}(tY)}) Y \mathrm{d}t.$$

Inserting the Taylor expansion for Ψ and integrating termwise one obtains the series formulation of the BCHD-formula $X * Y =$

$$X + \sum_{\substack{k,m \geq 0 \\ p_i + q_i > 0}} \frac{(-1)^k \mathrm{ad}(X)^{p_1} \mathrm{ad}(Y)^{q_1} \cdots \mathrm{ad}(X)^{p_k} \mathrm{ad}(Y)^{q_k} \mathrm{ad}(X)^m}{(k+1)(q_1 + \cdots + q_k + 1) p_1! q_1! \cdots p_k! q_k! m!} Y.$$

What we will prove instead...

1.9.2 Proposition (Toy BCDH-formula)

Let $X, Y \in M_n(\mathbb{K})$ be a pair of matrices with

$$[X, [X, Y]] = 0, \qquad [Y, [X, Y]] = 0.$$

Then

$$e^X \cdot e^Y = e^{X+Y+\frac{1}{2}[X,Y]}. \tag{1.15}$$

So in other words, in the situation of 1.9.2:

$$X * Y = X + Y + \frac{1}{2}[X, Y]$$

Local Lie groups

Why do we bother with BCDH?

1.9.8 Lemma

Let $(\mathfrak{g}, [\cdot, \cdot])$ be a finite dimensional Lie algebra. Then the BCHD-series $X * Y$ converges for $X, Y, Z \in \mathfrak{g}$ sufficiently near 0 and the following holds

$$(X * Y) * Z = X * (Y * Z)$$

as long as both sides are defined.

Upshot: Every Lie algebra encodes via BCHD a multiplication type operation.

1.9.7 Local Lie group

A *local Lie group* is an open 0-neighborhood $U \subseteq E$ in a finite dimensional vector space E such that there are smooth maps $\mu\colon U \times U \to E$ (*local multiplication*) and $\iota\colon U \to E$ (*inversion*), such that, for X, Y, Z sufficiently close to 0 in E the following holds

$$\mu(X,0) = X = \mu(0,X) \quad \mu(X,\iota(X)) = 0 = \mu(\iota(X),X)$$
$$\mu(X,\mu(Y,Z)) = \mu(\mu(X,Y),Z)$$

A *morphism of local Lie groups* is a smooth map $\phi\colon V \to U_2$ defined on an open 0-neighborhood such that there exists $0 \in W \subseteq V$ with $\mu_1(X,Y), \iota_1(X) \in V$ if $X, Y \in W$ and

$$\phi(\mu_1(X,Y)) = \mu_2(\phi(X),\phi(Y)), \quad \phi(\iota_1(X)) = \iota_2(\phi(X)).$$

1.9.9 Theorem (From Lie algebras to local Lie groups)

Every finite dimensional Lie algebra $\mathfrak{g}$ admits an open 0-neighborhood $U_{\mathfrak{g}}$ which becomes a local Lie group with respect to the mappings $\mu(x, y) = x * y$ defined via the BCDH-formula and $\iota(x) = -x$.

Further, every Lie algebra morphismn $\psi \colon \mathfrak{g} \to \mathfrak{h}$ gives rise to a morphism of local Lie groups $U_{\mathfrak{g}} \to U_{\mathfrak{h}}$.