# Introduction to Lie theory

Material for week 5

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# What we learned so far...

## 1.5.1 Definition

Let $G \subseteq \mathrm{Gl}_n(\mathbb{K})$ be a linear Lie group. Then we define

$$\mathbf{L}(G) := \{X \in M_n(\mathbb{K}) \mid \forall t \in \mathbb{R}, \exp(tX) \in G\}$$

and call this set the *Lie algebra* of $G$.

## Examples we already "computed"

With the commutator bracket of matrices (or linear maps):

1. $\mathfrak{gl}_n(\mathbb{K}) := \mathbf{L}(\mathrm{Gl}_n(\mathbb{K})) = M_n(\mathbb{K})$
2. $\mathfrak{gl}(V) := \mathbf{L}(\mathrm{Gl}(V)) = \mathrm{Lin}(V, V)$
3. $\mathbf{L}(\mathbb{K}^n) = \left\{ \begin{bmatrix} \mathbf{0} & x \\ 0 & 0 \end{bmatrix} : x \in \mathbb{K}^n \right\}$ abelian example. Last time the additive group $\mathbb{K}^n$ was realised as the linear Lie group $\begin{bmatrix} I_n & x \\ 0 & 1 \end{bmatrix}$.

# How can we compute the Lie algebra?

We know two methods:

**Using the matrix exponential**

Deduce from the group $G$ which matrices get exponentiated into $G$ (these then form $\mathbf{L}(G)$).

**1-parameter group method**

Due to Thm. 1.5.10 it suffices to find all 1-parameter groups $\gamma\colon \mathbb{R} \to G$ and compute their derivatives $\frac{d}{dt}\Big|_{t=0} \gamma(t)$

**Lemma 1.6.3 (Proof as Exercise!)**

Let $X \in M_n(\mathbb{K})$ and denote by $\operatorname{tr} X = \sum_{i=1}^{n} X_{ii}$ its trace. Then

$$\mathbf{d} \det(I_n)(X) = \operatorname{tr} X.$$

# Recall from Lie groups and geometry chapter...

$\beta\colon V \times V \to \mathbb{K}$ a bilinear form, $V$ finite dimensional vector space.

$$\mathrm{Aut}(V,\beta) := \{g \in \mathrm{Gl}(V)\colon (\forall v, w \in V)\ \beta(gv, gw) = \beta(v, w)\}$$

Recall $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{K})$ and $\mathrm{Lin}(V, V) \cong M_n(\mathbb{K})$.

Moreover, $\mathrm{Aut}(V,\beta)$ is a linear Lie group, whence we can compute

$$\mathfrak{aut}(V,\beta) := \mathbf{L}(\mathrm{Aut}(V,\beta))$$

# A technical Lemma

Note that via $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{K})$ and $\mathrm{Lin}(V, V) \cong M_n(\mathbb{K})$ we can pull back the matrix exponential to an exponential

$$\exp\colon \mathrm{Lin}(V, V) \to \mathrm{GL}(V)$$

Thus it makes sense to write $e^y$ for $y \in \mathrm{Lin}(V, V)$. To save space we also write for a linear map $y \in \mathrm{Lin}(V, V)$ and $v \in V$
$y.v := y(v)$.

**Lemma 1.6.6**

Let $V, W$ be finite dimensional vector spaces and $\beta\colon V \times V \to W$ a bilinear map. For $(x, y) \in \mathrm{Lin}(V, V) \times \mathrm{Lin}(W, W)$ the following are equivalent:

1. $e^{ty}.\beta(v, v') = \beta(e^{tx}.v, e^{tx}.v')$ for all $t \in \mathbb{R}$ and all $v, v' \in V$
2. $y.\beta(v, v') = \beta(x.v, v') + \beta(v, x.v')$ for all $v, v' \in V$.

## Cauchy-product formula for series

If $\beta \colon V \times V \to W$ is bilinear ($V, W$ finite dimensional vector spaces) and

$$v := \sum_{n=0}^{\infty} v_n, w := \sum_{n=0}^{\infty} w_n \in V$$

converge absolutely, then

$$\beta(v, w) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \beta(v_k, w_{n-k})$$

**Exercise**: Check `https://en.wikipedia.org/wiki/Cauchy_product` and adapt their proof to the above statement

# 1.7 The Lie group exponential and the unit component

# The Lie group exponential

## 1.7.1 Definition

Let $G$ be a linear Lie group. The restriction of the matrix exponential

$$\exp_G \colon \mathbf{L}(G) \to G$$

is called *Lie group exponential* of $G$.

## 1.7.2 Lemma

If $E, F$ are subvector spaces such that $E \oplus F = M_n(\mathbb{K})$, then the map

$$\Phi \colon E \times F \to \mathrm{Gl}_n(\mathbb{K}), (x, y) \mapsto \exp(x) \exp(y)$$

is smooth and there exists a $(0, 0)$ neighborhood on which $\Phi$ restricts to a diffeomorphism onto a $I_n$-neighborhood.

# The Lie group exponential is a local homeomorphism

## 1.7.3 Proposition

Let $G$ be a linear Lie group and $\exp_G$ its Lie group exponential. There exists an open 0-neighborhood $V_1 \subseteq M_n(\mathbb{K})$ and an open $I_n$-neighborhood $V_2$ such that $\exp_G$ restricts to a homeomorphism $V_1 \cap \mathbf{L}(G) \to V_2 \cap G$.

**Proof:** Pick open 0-neighborhood $U$ and $I_n$-neighborhood $W$ such that $\exp|_U^V$ is a diffeomorphism with inverse $\log_V$. Rest on the blackboard.

## The unit component

### 1.7.4 Lemma

Let $G$ be a topological group with unit $\mathbf{1}$, and define

$$G_0 := \{g \in G \mid \exists \gamma \colon [0,1] \to G \text{ continuous }, c(0) = \mathbf{1}, c(1) = g\}$$

the *path component of the unit* or *unit component*. Then $G_0$ is a normal subgroup (i.e. $gG_0g^{-1} = G_0$ for all $g \in G$).

### 1.7.5 Examples

The following Lie groups are connected, whence they coincide with their unit component: $\mathrm{Gl}_n(\mathbb{C}), \mathrm{U}_n(\mathbb{C}), \mathrm{SL}_n(\mathbb{C})$. However:

$$\mathrm{O}_n(\mathbb{R})_0 = \mathrm{SO}_n(\mathbb{R}), \qquad \mathrm{Gl}_n(\mathbb{R})_0 = \mathrm{Gl}_n(\mathbb{R})^+$$