Introduction to Lie theory

Material for week 4

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What we learned so far...

For $M \in M_n(\mathbb{K})$ we define the *matrix exponential*

$$e^M := \sum_{k=0}^{\infty} \frac{M^k}{k!}$$

We already know that for $X, Y \in M_n(\mathbb{K})$

- If $XY = YX$ then $e^{X+Y} = e^X e^Y$
- The map $\varphi_X \colon \mathbb{R} \to \mathrm{Gl}_n(\mathbb{K}), \varphi_X(t) = e^{tX}$ is a 1-parameter group.
- $\exp \colon M_n(\mathbb{K}) \to \mathrm{Gl}_n(\mathbb{K}), \exp(X) = e^X$ is smooth.
- We can find (arbitrarily small) neighborhoods U of 0 and V I_n such that $\exp|_U^V$ is a diffeomorphism.

The matrix logarithm

Definition 1.4.10

If U, V is a pair of neighborhoods such that exp restricts to a diffeomorphism on them, we set

$$\log_V \colon V \to U, \quad g \mapsto (\exp|_U)^{-1}(g)$$

and call this function the *matrix logarithm* on V. We will write also log if the set V is not important for the argument.

One can show that

$$\log_V(g) = \log_V(I_n + (g - I_n)) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(g - I_n)^k$$

however, we shall not use this identity.

All 1-parameter groups are smooth!

Recall that we defined a 1-parameter group as a **continuous** group morphism $\gamma\colon (\mathbb{R},+) \to \mathsf{Gl}_n(\mathbb{K})$.

We shall now show that these morphisms are indeed smooth!

Theorem 1.4.11 (1-parameter group theorem)

Every 1-parameter group γ is of the form $\gamma(t) = e^{tX}$. So γ is smooth and completely determined by its derivative in 0.

Material from Appendix A.2

The mean value theorem in integral form

Vector valued integrals for $f\colon [a,b] \to \mathbb{R}^n$ with $f = (f_i)$

$$\int_a^b f(t)\mathrm{d}t := \begin{bmatrix} \int_a^b f_1(t)\mathrm{d}t \\ \vdots \\ \int_a^b f_N(t)\mathrm{d}t \end{bmatrix}$$

Proposition A.2.10

Let $U \subseteq \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}^d$ be a C^1-map. Assume that for $x, y \in U$ also the line segment $\overline{xy} := \{tx + (1-t)y\colon t \in [0,1]\} \subseteq U$, then

$$f(y) - f(x) = \int_0^1 \mathbf{d}f(x + t(y-x))(y-x)\mathrm{d}t.$$

In words: $f(y) - f(x)$ can be computed by integrating $\mathbf{d}f$ along $\overline{xy}$

Parameter integrals depend continuously on the parameter

Lemma A.2.9

Let $a < b$ be real numbers, $U \subseteq \mathbb{R}^n$ be open and $f\colon [a, b] \times U \to \mathbb{R}^m$ continuous. Then

$$F\colon U \to \mathbb{R}^m, \quad F(u) := \int_a^b f(t, u)\mathrm{d}t$$

is continuous. In particular,

$$\lim_{u \to u_0} F(u) = \int_a^b \lim_{u \to u_0} f(t, u)\mathrm{d}t.$$

1.5 The Lie algebra of a linear Lie group

The Lie algebra via the exponential

1.5.1 Definition

Let $G \subseteq \mathrm{Gl}_n(\mathbb{K})$ be a linear Lie group. Then we define

$$\mathbf{L}(G) := \{X \in M_n(\mathbb{K}) \mid \forall t \in \mathbb{R}, \exp(tX) \in G\}$$

and call this set the *Lie algebra* of G.

1.5.2 Lemma

The Lie algebra of a linear Lie group G is an $\mathbb{R}$-vector space.

1.5.3 Remark

In general $\mathbf{L}(G)$ will not be closed under multiplication with complex numbers, so it is not a $\mathbb{C}$-vector space.

Tools and definitions

Recall the (Lie-)Trotter product formula

$$\exp(X+Y) = \lim_{k\to\infty} \left(e^{X/k} e^{Y/k}\right)^k$$

1.5.6 Definition (Abstract Lie algebra)

Let L be a $\mathbb{K}$-vector space. A $\mathbb{K}$-bilinear map $[\cdot,\cdot]\colon L \times L \to L$ is called a *Lie bracket* (over $\mathbb{K}$) if

(L1) $[x,x] = 0$ for $x \in L$ and

(L2) $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$ for $x,y,z \in L$ (Jacobi identity)

A *Lie algebra* (over $\mathbb{K}$) is a $\mathbb{K}$-vector space L, endowed with a Lie bracket. A subspace $E \subseteq L$ of a Lie algebra is called a *subalgebra* if $[E,E] \subseteq E$. A Lie algebra is said to be *abelian* of $[x,y] = 0$ holds for all $x,y \in L$.

(Homo-)morphisms of Lie algebras

A *homomorphism* $\varphi\colon L_1 \to L_2$ of Lie algebras is a linear map with $\varphi([x,y]) = [\varphi(x), \varphi(y)]$ for $x, y \in L_1$.

Abelian Lie algebra

Every $\mathbb{K}$-vector space is an abelian Lie algebra with the trivial Lie bracket.

The cross product (Exercise)

$\mathbb{R}^3$ is a Lie algebra with respect to the cross product

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} := \begin{bmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{bmatrix}$$