Introduction to Lie theory

Material for week 3

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What we learned so far...

The main example

The group $\mathrm{Gl}_n(\mathbb{K})$ of all invertible $n \times n$-matrices is a Lie group.

We motivated our studies by looking at :

1-parameter groups of symmetries of $O \subseteq \mathbb{R}^n$

An infinite family of symmetries $\Gamma_\varepsilon, \varepsilon \in \mathbb{R}$ (i.e. the Γ_ε preserve O) a 1-parameter group if

- $\Gamma_0 = \mathrm{id}_{\mathbb{R}^n}$ is the trivial symmetry
- $\Gamma_\varepsilon \circ \Gamma_\delta = \Gamma_{\varepsilon+\delta}$
- The map $\Gamma\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, (\vec{x}, \varepsilon) \mapsto \Gamma_\varepsilon(\vec{x})$ is smooth.

One parameter groups of matrices

Definition 1.4.1

Let $G \subseteq \mathrm{Gl}_n(\mathbb{K})$ be a linear Lie group A 1-*parameter group in G* is a continuous group morphism $\varphi\colon (\mathbb{R}, +) \to G$.

Note that this implies

$$\varphi(t+s) = \varphi(t)\cdot\varphi(s)(= \varphi(s)\cdot\varphi(t)).$$

In particular, $\varphi(0) = I_n$. The action map

$$\Gamma\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, (\vec{x}, t) \mapsto \varphi(t)\vec{x}$$

is smooth in $\vec{x}$ (Exercise!) We will see later that it automatically smooth in t.

How do we find examples of such groups?

1.4.2 Example

For $n = 1$, we have $\mathrm{Gl}_n(\mathbb{R}) = \mathbb{R}^\times := \mathbb{R} \setminus \{0\}$ and a smooth map

$$\exp\colon \mathbb{R} \to \mathrm{Gl}_1(\mathbb{R}), \exp(t) = e^t.$$

Due to the familiar properties of the exponential function, this map is a (smooth!) 1-parameter group in $\mathrm{Gl}_1(\mathbb{R})$.

Idea: Can we generalise this by generalising the exponential function

$$e^z = \sum_{k=0}^{\infty} \frac{z^n}{n!}$$

This sum only uses multiplication and addition (and convergence), all of this should probably work for matrices.

The matrix exponential function

The matrix exponential function

For $M \in M_n(\mathbb{K})$ we define the *matrix exponential*

$$e^M := \sum_{k=0}^{\infty} \frac{M^k}{k!}$$

We have to check that this thing actually converges:

$$S_\ell := S_\ell(M) := \sum_{k=1}^{\ell} \frac{M^k}{k!}$$

Using that $\|A \cdot B\| \leq \|A\| \cdot \|B\|$, we get for $\varepsilon > 0$ that

$$\|S_\ell - S_m\| \leq \sum_{k=m+1}^{\ell} \left\| \frac{M^k}{k!} \right\| \leq \sum_{k=m+1}^{\ell} \frac{\|M\|^k}{k!} < \varepsilon \text{ for } \ell, m \text{ large,}$$

Hence we have a Cauchy-sequence and the series converges for every matrix M to some matrix e^M. Note

$$\|e^M\| \leq e^{\|M\|}$$

A.3 Differential equations and vector fields

A.3.1 Remark

Let $U \subseteq \mathbb{R}^n$ be open for some $n \in \mathbb{N}$. A function $F \in C^k(U, \mathbb{R}^n)$ is also called a C^k-*vector field* on U.

We want to solve differential equations of the form

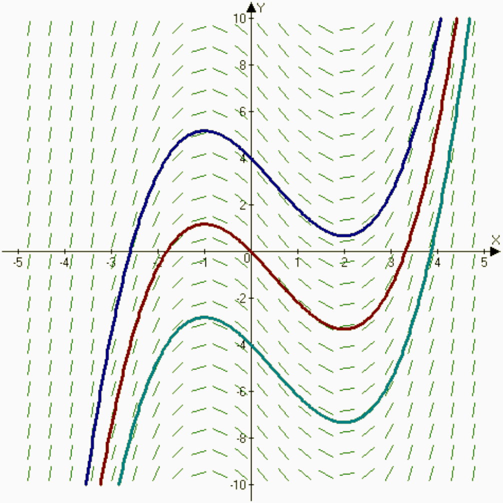
$$\gamma'(t) := \frac{d}{dt}\gamma(t) = X(\gamma(t)), \qquad X \in C^k(U, \mathbb{R}^n)$$

A.3.2 Definition

Let I be an open interval containing 0 and $\gamma\colon I \to U \subseteq \mathbb{R}^n$ be a differentiable map. We call γ and *integral curve* of a vector field $X \in C^k(U, \mathbb{R}^n)$ if

$$\gamma'(t) = X(\gamma(t)) \text{ for each } t \in I. \tag{1}$$

Vector fields and integral curves



From Wikipedia

`https://en.wikipedia.org/wiki/Integral_curve`

The local existence and uniqueness theorem

A.3.4 Definition

For $\gamma\colon (a,b) \to U \subseteq \mathbb{R}^n$ continuous, write

$$\lim_{t\to b} \gamma(t) = \infty$$

if for each compact $K \subset U$ there is $c \in (a,b)$ such that for all $t > c$ we have $\gamma(t) \notin K$.

Similarly we write $\lim_{t\to a} \gamma(t) = \infty$ if for each $K \subseteq U$ compact there is $c \in (a,b)$ with $\gamma(t) \notin K$ for all $t < c$.

A.3.5 Local Existence and Uniqueness result

Let $X \in C^k(U, \mathbb{R}^n), k \geq 1$ for $U \subseteq \mathbb{R}^n$ open, $p \in U$. Then there exists a unique maximal integral curve $\gamma_p\colon I_p \to U$ with $\gamma_p(0) = p$. If $a := \inf I_p > -\infty$, then $\lim_{t\to a} \gamma_p(t) = \infty$ and if $b := \sup I_p < \infty$, then $\lim_{t\to b} \gamma_p(t) = \infty$.

Integral curves and flows

Note that if X is C^k, also

$$\gamma'(t) = X(\gamma(t))$$

is continuous even C^{k+1}.

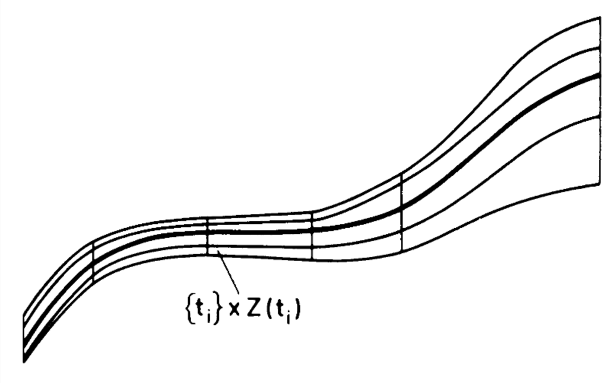
If $J \supseteq I$ is an interval containing I, then an integral curve $\eta\colon J \to U$ of X is called an *extension* og γ if $\eta|_I = \gamma$. An integral curve γ is said to be *maximal* if it has no proper extensions.

A.3.8 Definition

A vector field $C^k(U, \mathbb{R}^n)$ is called *complete* if for every $p \in U$ the maximal integral curve γ_p is defined on $I_P = \mathbb{R}$

Local flows of vector fields

Considering all integral curves at the same time



From Amann, H: Ordinary differential equations, 1990

Local flows

Definition A.3.9

Let $U \subseteq \mathbb{R}^n$ open. A *local flow* on U is a smooth map $\Phi\colon \Omega \to U$, where $\Omega \subseteq \mathbb{R} \times U$ is open and for each $x \in U$ the intersection $I_x := \Omega \cap (\mathbb{R} \times \{x\})$ is an interval containing 0 and

$$\Phi(0, x) = x \text{ and } \Phi(t, \Phi(s, x)) = \Phi(t + s, x), \quad \forall t, s, x$$

for which both sides of the equation are defined.

The maps $\alpha_x\colon I_x \to U, t \mapsto \Phi(t, x)$ are called *flow lines*. The flow Φ is said to be *global* if $\Omega = \mathbb{R} \times U$.

A.3.12 Theorem

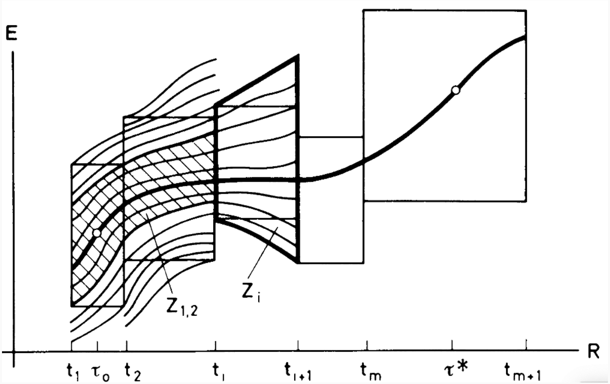
Each $X \in C^\infty(U, \mathbb{R}^n)$ is the velocity field of a unique local flow $\mathrm{Fl}^X \colon \mathcal{D}_X \to \mathbb{R}^n$ defined by

$$\mathcal{D}_X := \bigcup_{x \in U} I_x \times \{x\} \text{ and } \mathrm{Fl}^X(t, x) := \gamma_x(t) \text{ for } (t, x) \in \mathcal{D}_X,$$

where $\gamma_x \colon I_x \to U$ is the unique maximal integral curve through $x \in U$.

The flow takes intial conditions and builds integral curves starting at these initial conditions.

Integral curves smoothly varying



From Amann: ODE, 1990

Note: We will not give the proofs here! In case you want to know more consult the literature.

Local flows are smooth in all parameters

Let $U \subseteq \mathbb{R}^n$ and $P \subseteq \mathbb{R}^k$ be open. Assume that $\Psi\colon P \to C^\infty(U, \mathbb{R}^n)$ is a map such that the map

$$\hat{\Psi}\colon P \times U \to \mathbb{R}^n, (p, x) \mapsto \Psi(p)(x)$$

is smooth. Assume that all the vector fields $\Psi(p)$ are complete. Then

Global version of Theorem A.3.13

There exists a smooth map $\Phi\colon \mathbb{R} \times P \times U \to U$ such that for each $(p, x) \in P \times U$ the curve

$$\Phi_x^p\colon \mathbb{R} \to U, \quad t \mapsto \Phi(t, p, x)$$

is an integral curve of the vector field $\Psi(p)$ with $\Phi_x^p(0) = x$.

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is smooth. Now we do not assume that the vector fields are compact. Then

A.3.13 Theorem

For each $(p_0, x_0) \in P \times U$ exists an open neighborhood W of p_0 in P, an open interval $0 \in I \subseteq \mathbb{R}$, an open neighborhood $x_0 \in V \subseteq U$ and a smooth map $\Phi\colon I \times W \times V \to U$ such that for each $(p, x) \in W \times V$ the curve

$$\Phi_x^p\colon I \to U, \quad t \mapsto \Phi(t, p, x)$$

is an integral curve of the vector field $\Psi(p)$ with $\Phi_x^p(0) = x$.