Introduction to Lie theory

Recap of material for week 2

Alexander Schmeding

26. August 2024

What we learned so far...

Let G be a group such that...

- G is also a topological space and the group operations $m\colon G \times G \to G$ and $i\colon G \to G$ are continuous. Then G is a *topological group*.
- G is also a manifold[1] and the group operations $m\colon G \times G \to G$ and $i\colon G \to G$ are smooth. Then G is a *Lie group*.

The main example

The group $\mathrm{Gl}_n(\mathbb{K})$ of all invertible $n \times n$-matrices is a Lie group.

[1] Open subsets of euclidean spaces are manifolds

Linear Lie groups

All closed subgroups of $\mathrm{Gl}_n(\mathbb{K})$ are called *linear Lie groups*.

We know the following examples

$\mathrm{Sl}_n(\mathbb{K})$, $(\mathrm{S})\mathrm{O}_n(\mathbb{K})$, $(\mathrm{S})\mathrm{U}_n(\mathbb{K})$

These are all topological groups as subgroups of the topological group $\mathrm{Gl}_n(\mathbb{K})$.

Lemma B.0.1

Let G be a topological group and H a subgroup. Then the subspace topology turns H into a topological group.

Chapter 1.2: Topological structure of matrix Lie groups

Topological properties I

If $(X, \mathcal{T})$ is a topological space, we say that X is *compact*, if every open cover $(U_i)_{i \in I}$ has a finite subcover, i.e. $X = U_{i_1} \cup \cdots \cup U_{i_n}$ for some $n \in \mathbb{N}$.

More useful for us:

Heine-Borel theorem

A subset $K \subseteq \mathbb{R}^n$ is compact if and only if

- K is **bounded**[2]
- K is *closed*.

[2] A set is bounded if $K \subseteq B_r(0)$ for some $r > 0$, i.e. it is contained in some sufficiently large ball.

Topological properties II

A topological space $(X, \mathcal{T})$ is called *path connected* if for every pair $x, y \in X$ there is a continuous map $c\colon [0,1] \to X$ with $c(0) = x$ and $c(1) = y$ (a path from x to y).

If $x \in X$ and $U_x \subseteq X$ is the maximal set which contain x and is path connected, we call U_x the *path component* of x in X.

You should think of the path component as all points we can reach using continuous paths starting from x.

For those who know topology

For subsets of euclidean space path connectedness is equivalent to connectedness (i.e. that the space can not be partitioned in disjoint open and closed subsets).

1.3 Groups and geometry

Abstract vector spaces

Definition 1.3.1

Let V be a $\mathbb{K}$-vector space. Write $\mathrm{Gl}(V)$ for the group of all invertible linear maps in $\mathrm{Lin}(V, V)$ (vector space automorphisms)

If V is n-dimensional and $v_1, \ldots, v_n$ a basis, then

$$\Phi\colon M_n(\mathbb{K}) \to \mathrm{Lin}(V, V), \quad \Phi(A)v_k := \sum_{j=1}^{n} A_{jk} v_j$$

is a linear ismorphism. It describes the linear map associated to a matrix, its inverse computes the matrix representing a linear map in the chosen basis)

We now have two identifications $\mathrm{Lin}(V, V) \cong M_n(\mathbb{K}) \cong \mathbb{K}^{n^2}$

A group isomorphism

Note $\Phi(I_n) = \mathrm{id}_V$ and $\Phi(A \cdot B) = \Phi(A) \circ \Phi(B)$. So Φ restricts to an isomorphism (of groups!)

$$\Phi' \colon \mathsf{Gl}_n(\mathbb{K}) \to \mathsf{Gl}(V).$$

We can thus later also work with the Lie group $\mathsf{Gl}(V)$ (its Lie group structure is the one we obtain by identifying it with $\mathsf{Gl}_n(\mathbb{K})$)

Bilinear forms

Recall that a bilinear map $\beta\colon V \times V \to \mathbb{K}$ satisfies
$\beta(cx + y, v) = c\beta(x, v) + \beta(y, v)$ and
$\beta(x, cv + w) = c\beta(x, v) + \beta(x, w)$.

We construct a matrix for each bilinear formel

$$B = (B_{ij}) := (\beta(v_j, v_k))_{j,k=1,\dots,n}$$

Then it is easy to see that if we identify via the basis V with $\mathbb{K}^n$ that

$$\beta(x, y) = x^\top By \quad \text{matrix multiplication!}$$

So B does not behave like the representation matrix of a linear map!