Introduction to Lie theory

Material for week 13

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4. Lie group actions

Lie group actions

A smooth map $\sigma\colon G \times M \to M$ (G Lie group, M manifold) is called a **Lie group action** if it satisfies

(A1) $\sigma(\mathbf{1}_G, m) = m$ for all $m \in M$

(A2) $\sigma(g_1, \sigma(g_2, m)) = \sigma(g_1 g_2, m)$ for all $g_1, g_2 \in G, m \in M$

We write shorter

$$\sigma_g(m) := \sigma^m(g) := g.m := \sigma(g, m)$$

4.0.2 Left and right actions

What we call an action is also called **left action**.

A (smooth) **right action** is a smooth map $\sigma_R \colon M \times G \to M$ such that for all $g_1, g_2 \in G$ and $m \in M$

$$\sigma_R(m, \mathbf{1}_G) = m, \quad \sigma_R(\sigma_R(m, g_1), g_2) = \sigma_R(m, g_1 g_2)$$

If σ_R is a right action then

$$\sigma(g, m) := \sigma_R(m, g^{-1})$$

is a left action. A similar formula transforms left into right actions. Since there is no essential difference we focus on left actions

4.1 Orbits and stabilisers

Orbits and orbit decomposition

4.1.1 Definition

Let $\sigma \colon G \times M \to M$ be a Lie group action. For $m \in M$, the set

$$\mathcal{O}_m := G.m := \{g.m \colon g \in G\} = \{\sigma(g, m) \colon g \in G\}$$

is called the *orbit of m*.The action is said to be *transitive* if there exists only one orbit, i.e., for $x, y \in M$, there exists $g \in G$ with $y = g.x$.We write $M/G := \{\mathcal{O}_m \colon m \in M\}$ for the set of G-orbits on N

4.1.2 Remark

The orbits form a partition of M (Exercise).

A subset $R \subseteq M$ is called a set of *representatives for the action* if each G-orbit in M meets R exactly once: $\forall x \in M, |R \cap \mathcal{O}_x| = 1$.

4.1.6 Definition

Let $\sigma\colon G \times M \to M$ be an action of the group G on M. For $m \in M$, the subset

$$G_m := \{g \in G\colon g.m = m\}$$

is called the *stabiliser of* m.For $g \in G$, we write

$$\mathrm{Fix}(g) := M^g := \{m \in M\colon g.m = m\}$$

for the *set of fixed points of* g in M. We then have

$$m \in M^g \Leftrightarrow g \in G_m.$$

For subset $S \subseteq M$ and $H \subseteq G$ we write

$$G_S := \bigcap_{m \in S} G_m = \{g \in G\colon (\forall m \in S) g.m = m\},$$
$$M^H := \{m \in M\colon (\forall h \in H)\ h.m = m\}.$$

4.2 Homogeneous spaces

4.2.1 Definition

Let G be a group and H a subgroup. We write

$$G/H := \{gH : g \in G\}$$

for the *set of left cosets of H* in G and $q_{G/H}\colon G \to G/H, g \mapsto gh$ for the quotient map.Then

$$\sigma\colon G \times G/H \to G/H, \quad (g, xH) \mapsto gxH$$

defines a transitive action of G on the set G/H.

Equivariant maps

4.2.2 Definition

Let G be a group and $\sigma_i \colon G \times M_i \to M_i, i = 1, 2$ two actions of the group G on sets. A map $f \colon M_1 \to M_2$ is called *G-equivariant* if

$$f(g.m) = g.f(m) \quad \text{holds for all } g \in G, m \in M_1.$$

Idea: For a group action $\sigma \colon G \times M \to M$ the map

$$G/G_m \to \mathcal{O}_m, \quad gG_m \mapsto g.m$$

is equivariant. We will now construct a manifold structure on G/H and transport it via the equivariant map to the orbit $\mathcal{O}_m$!