Introduction to Lie theory

Material for week 11

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3.1 The Lie algebra of a Lie group

Lie groups and their algebras

A *Lie group* G is a manifold G endowed with a group structure such that the multiplication map $m_G\colon G \times G \to G$ and the inversion map $\iota\colon G \to G$ are smooth. A vector field X on G is called left-invariant if

$$T\lambda_g \circ X = X \circ \lambda_g \quad g \in G.$$

The Lie algebra of G

$\mathbf{L}(G) = T_{\mathbf{1}}G$ with Lie bracket induced by the linear isomorphism

$$\Theta\colon T_{\mathbf{1}}G \to \mathcal{V}^{\ell}(G), \quad \Theta(v) = (g \mapsto T\lambda_g(v))$$

We write shorter $v_\ell := \Theta(v)$ for the left-invariant vector field, then the Lie bracket is defined as $[v, w] = [v_\ell, w_\ell](\mathbf{1})$

Examples

3.1.4 Example

The Lie algebra of the abelian Lie group $(\mathbb{R}^n, +)$ is the abelian Lie algebra $\mathbb{R}^n$.

3.1.5 Example

The linear Lie group $\mathrm{GL}_n(\mathbb{K})$ is open in $M_n(\mathbb{K})$, SO $T_{\mathbf{1}}\mathrm{GL}_n(\mathbb{K}) = M_n(\mathbb{K}) = \mathfrak{gl}_n(\mathbb{K})$.We already computed the left invariant vector field associated to $X \in M_n(\mathbb{K})$:

$$X_\ell(A) = (A, AX), A \in \mathrm{GL}_n(\mathbb{K}).$$

Hence with $\tilde{X}_\ell(A) = AX$ the Lie bracket turns out to be

$$[X, Y] = [X_\ell, Y_\ell](I_n) = \mathbf{d}\tilde{Y}_\ell(I_n)(X) - \mathbf{d}\tilde{X}_\ell(I_n)(Y) = YX - XY$$

Hence we recover the commutator bracket!

From Lie group morphism to Lie algebra morphism

Recall that for a morphism $\varphi\colon G_1 \to G_2$ of linear Lie groups we constructed a Lie algebra morphism

$$\mathbf{L}(\varphi)(x) := \left.\frac{d}{dt}\right|_{t=0} \varphi(\exp_{G_1}(tx))$$

Since the Lie group exponential is just the matrix exponential we saw that it induces a chart around the identity.

So the above formula tries to compute the derivative of φ at $\mathbf{1}$!

We can emulate this (without the matrix exponential) as $\mathbf{L}(\varphi) = T_{\mathbf{1}}\varphi$

3.2 The Lie group exponential

3.2.2 Definition

Define the *Lie group exponential* for a Lie group G as

$$\exp_G \colon \mathbf{L}(G) \to G, \quad \exp_G(x) := \gamma_x(1),$$

where $\gamma_x \colon \mathbb{R} \to G$ is the unique maximal integral curve of the left invariant vector field x_ℓ, satisfying $\gamma_x(0) = \mathbf{1}$. This means that γ_x is the unique solution of the initial value problem

$$\gamma(0) = \mathbf{1}, \quad \frac{d}{dt}\gamma(t) = x_\ell(\gamma(t)) = T_{\mathbf{1}}\lambda_{\gamma(t)}(x), \quad \forall t \in \mathbb{R}.$$

3.2.3 Example $G = \mathbf{Gl}_n(\mathbb{K})$

Left invariant vector fields are of the formul

$$X_\ell(A) = (A, AX), \text{ i.e. the ODE reads } \dot\gamma(t) = \gamma(t)X$$

which is solved by $\gamma(t) = e^{tX}$ i.e. Lie group exponential is the matrix exponential.

3.2.4 Lemma (Properties of the exponential)

Let G be a Lie group. For each $x \in \mathbf{L}(G)$,

1. the curve $\gamma_x \colon \mathbb{R} \to G, \gamma_x(t) = \exp_G(tx)$ is a smooth homomorphism of Lie groups with $\gamma_x'(0) = x$.
2. The global flow of the left invariant vector field x_ℓ is given by

$$\mathrm{Fl}^{x_\ell}(t, g) = g\gamma_x(t) = g \exp_G(tx).$$

3. If $\gamma \colon \mathbb{R} \to G$ is a smooth homomorphism of Lie groups and $\dot{\gamma}(0) = x$, then $\gamma = \gamma_x$. In particular, the map $\mathrm{Hom}(\mathbb{R}, G) \to \mathbf{L}(G), \gamma \mapsto \dot{\gamma}(0)$ is a bijection (here $\mathrm{Hom}(\mathbb{R}, G)$ denotes the set of smooth homomorphisms of Lie groups $\mathbb{R} \to G$.

3.2.5. Naturality of $\exp_G$

Let $\varphi\colon G_1 \to G_2$ be a morphism of Lie groups and $\mathbf{L}(\varphi)\colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ its differential at $\mathbf{1}$. Then

$$\exp_{G_2} \circ \mathbf{L}(\varphi) = \varphi \circ \exp_{G_1} .$$

3.2.6. Lemma

For a Lie group G, the Lie group exponential $\exp_G\colon \mathbf{L}(G) \to G$ is smooth and satisfies $T_0 \exp_G = \mathrm{id}_{\mathbf{L}(G)}$.

In particular, $\exp_G$ is a local diffeomorphism at 0 in the sense that it maps some 0-neighborhood in $\mathbf{L}(G)$ diffeomorphically onto some $\mathbf{1}_G$-neighborhood in G.

3.2.8 Proposition (Lie-Trotter formula)

Let G be a Lie group and $x, y \in \mathbf{L}(G)$ then

$$\exp_G(x+y) = \lim_{k\to\infty} \left(\exp_G\left(\frac{x}{k}\right)\exp_G\left(\frac{y}{k}\right)\right)^k .$$

3.2.9 Canonical coordinates

Let G be a Lie group and $b_1, \ldots, b_n$ be a basis for $\mathbf{L}(G)$. Then the following maps restrict to diffeomorphisms of some 0-neighborhood in $\mathbb{R}^n$ to some open $\mathbf{1}_G$-neighborhood in G:

(i) $x \mapsto \exp_G(x_1 b_1 + \cdots + x_n b_n)$ (coordinates of the first kind)

(ii) $x \mapsto \exp_G(x_1 b_1) \cdots \exp_G(x_n b_n)$ (coordinates of the second kind).