Introduction to Lie theory

Material for week 9

Alexander Schmeding

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Reklame: Onsager Lecture ´24

Martin Hairer: Taming infinities

Abstract: Some physical and mathematical theories have the unfortunate feature that if one takes them at face value, many quantities of interest appear to be infinite! What's worse, this doesn't just happen for some exotic pieces of abstract mathematics, but in the standard theories describing some of the most fundamental aspects of nature. ...

Thursday 24.10.24, kl. 14.15, S8

`https://www.ntnu.edu/onsager`

3. Lie groups beyond matrix groups

3.0.1 Definition

A *Lie group* G is a manifold G endowed with a group structure such that the multiplication map $m_G \colon G \times G \to G$ and the inversion map $\iota \colon G \to G$ are smooth. A morphism of Lie groups is a smooth group homomorphism.

3.0.2 Example

For every $d \in \mathbb{N}_0$ every vector space $\mathbb{R}^d$ is a Lie group. Every linear map $f \colon \mathbb{R}^d \to \mathbb{R}^n$ is Lie group morphism.

3.0.3 Example (Consequence of Proposition 1.1.2)

For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ the group $\mathrm{GL}_n(\mathbb{K})$ is a Lie group.

Standard Notation

Let G be a Lie group, we shall write

- $\mathbf{1}_G$ for the unit element (or shorter $\mathbf{1}$),
- m_G for multiplication, ι_G for inversion,
- for $g \in G$ we let $\lambda_g \colon G \to G, h \mapsto gh$ and $\rho_g \colon G \to G, h \mapsto hg$ the *left (right) translation*.

(Observe that $\lambda_g(\rho_h(x)) = gxh = \rho_h(\lambda_g(x))$.)

3.0.4. Example (Circle and torus group)

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

is a submanifold of $\mathbb{R}^2$. Identifying it with $\{z \in \mathbb{C} \mid |z| = 1\}$, it also inherits a group structure, given by

$$(x, y) \cdot (x', y') := (xx' - yy', xy' + x'y) \text{ and } (x, y)^{-1} = (x, -y).$$

This structure turns the circle into a Lie group.

Then the torus $\mathbb{T}^n := (\mathbb{S}^1)^n$ is also a Lie group with

$$(t_1, \ldots, t_n) \cdot (s_1, \cdots, s_n) = (t_1 s_1, \ldots, t_n s_n).$$

Exercise: Check the details!

Appendix C: The Heisenberg quotient is not linear

The Heisenberg quotient (Example 3.0.8)

The Lie group

$$Q := \mathbb{R}^2 \times \mathbb{S}^1 \text{ denote elements } ((x,y),z) \in \mathbb{R}^2 \times \mathbb{S}^1$$

with multiplication

$$((x,y),z) \cdot ((a,b),c) = (x+a, y+b, zce^{\mathrm{i}xb})$$

C.0.1 Proposition

Q is not isomorphic to a subgroup of $\mathrm{GL}_n(\mathbb{K})$, $n \in \mathbb{N}$.

Note: Since we can embedd

$$\mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{C}), A \mapsto A$$

as a (closed) subgroup, it suffices to prove C.0.1 for $\mathbb{K} = \mathbb{C}$

Two technical lemmata I

C.0.2 Lemma

Let $X \in M_n(\mathbb{C})$ and $p \in \mathbb{N}$ such that $X^p = I_n$. Then X is diagonalisable over $\mathbb{C}$ and all eigenvalues of X are pth roots of unity, i.e. $\lambda^p = 1$.

Jordan normal form

For $X \in M_n(\mathbb{C})$ there are complex linear subspaces $\mathbb{C}^n = E_1 \times \cdots \times E_i$, $i \in \mathbb{N}$ such that $X(E_j) \subseteq E_j$ and

$$X|_{E_j} = \lambda_j I_j + N_j \text{ where } N_j \text{ is strictly upper triangular}$$

Two technical lemmata II

C.0.3 Lemma

Let $n \in \mathbb{N}$ and p a prime number. Assume that there are $A, B, C \in M_n(\mathbb{C})$ with the following properties:

- $ABA^{-1}B^{-1} = C$
- $[A, C] = 0 \quad [B, C] = 0,$
- $C^p = I_n$ and p is the smallest number with this property.

Then $n \geq p$.

Proof of C.0.1: Let $p \in \mathbb{N}$ be any prime number. Consider $a = ((1, 0), 1)$, $b = (0, \frac{2\pi}{p}, 1)$ and

$$c = aba^{-1}b^{-1} = (0, 0, e^{\frac{2\pi \mathrm{i}}{p}}).$$

Then $c^p = (0, 0, 1)$ (the unit in Q!).