Algebraic topology I

A brief survey of basic concepts and results, fall semester 2012, NTNU

- the course builds on basic results of general topology and basic algebra (groups, rings, etc.)

- thus we take for granted the standard theory about the category of topological space and continuous maps.

I However, closely related to this is the homotopy theory, namely the category where a topological space is only "recognized" by its homotopy type and similarly, continuous maps f ~~are~~ replaced by their homotopy type $[f]$.

You should be able to answer questions such as:

a) What is the meaning of $f \sim g$ (homotopy between functions)?

b) X and Y are homotopy equivalent spaces? Means?

c) homotopy of maps $f, g : (X, A) \longrightarrow (Y, B)$?

where $A \subset X$, $B \subset Y$, etc. ...

d) retractions, deformation retraction, ...

d) A special case is the fundamental group $\pi_1(X, x_0)$ of space X with base point x_0. Give the precise definition of all ingredients, and explain how $\pi_1(X, x_0)$ becomes a group. Also the functorial properties of the construction $(X, x_0) \longrightarrow \pi_1(X, x_0)$

e) About <u>covering spaces</u> : $\tilde{X} \xrightarrow{p} X$,
This is Chapter 4 of the text book (cf. also the handwritten note about covering spaces and the <u>groups</u> <u>involved</u>. Questions :

1) What is the definition of a covering space ?
2) What is a regular (= normal) covering space ?
3) universal covering ?

etc.

II <u>Singular homology</u>, and the first homotopy group $\pi_1(X)$

Basic concepts are :

singular p-simplex, the (singular) chain complex, boundary operator ∂, homology groups $H_k(X)$, induced homomorphism $f_*: H_k(X) \longrightarrow H_k(Y)$, ~~the~~, the long exact homology sequence of a pair (X, A),

In what sense is the following statement true :

$\{H_k, k \geq 0\}$ are functors from the homotopy category to the category of groups !

– the <u>Mayer-Vietoris</u> exact sequence of $X = X_1 \cup X_2$.
What are the assumptions ?
Applications ?

What is $H_k(\mathbb{R}^n)$, $H_k(S^n)$, other examples ?
$k \geq 0$ $\quad$ $k \geq 0$

~~[illegible]~~ =

- Various concepts, applications of the homology functors H_k, or also homotopy, and the group $\pi_1(X)$.

- Hurewicz homomorphism : $\pi_1(X, x_0) \xrightarrow{h} H_1(X)$
Define the map h. What is the basic result about the connection between $\pi_1(X)$ and $H_1(X)$?
- What is the degree of a map $f: S^n \to S^n$?
- Is there any vector field on S^{2n} without zero points (fixpoints) ?
- Brouwer's fixed-point theorem for maps $f: D^n \to D^n$.

III Constructions of spaces (cf. Chap. 2)

CW-complex,
attaching an n-cell to a space Y, get Y_f.

More generally, $\left.\begin{array}{c} A \xrightarrow{f} Y \\ \cap \\ X \end{array}\right\}$ What is $X \cup_f Y$?

- If $Y_f = Y \cup_f D^n$, how to apply Mayer-Vietoris to calculate the homology of Y_f from that of Y, using an exact sequence ? (see Prop. 2.3)

- For a finite CW-complex X, there is a chain complex

$$\longrightarrow C_k(X) \xrightarrow{\partial} C_{k-1}(X) \xrightarrow{\partial}$$

where each $C_k(X)$ is free abelian (as with the singular chain complex) and finitely generated, from which one can calculate the homology of X, as usual. Explain roughly the construction of the chain complex.

- describe a simple way to build $\mathbb{C}P^n$ as a CW-complex ~~with a~~. Can ~~the~~ you calculate $H_*(\mathbb{C}P^n)$ using this?
- What about $\mathbb{R}P^n$? (smallest number of cells)
- For $X = S^n$, there is a very simply way to build S^n as a CW-complex. Use this to calculate $H_*(S^n)$ in a very simple way!

- For a finite CW-space X and subcomplex A. Can you say something about the connection between the homology ~~of~~ $H_*(X, A)$ and $H_*(X/A)$? Do you need any extra condition on $A \subset X$? Or is the appropriate condition automatic for pairs of CW-spaces?

- What is the <u>Euler characteristics</u> of X? If X is a CW-complex with only finitely many cells, how to compute this number in an easy way?

- We have that $\pi_1(X \times Y) \simeq \pi_1(X) \times \pi_1(Y)$. How would you construct the desired isomorphism (in a natural-obvious-way)?

IV <u>Some special topics</u> (cf. also a little in chap. 3)

- dont forget van <u>Kampen theorem</u>, What does it say? (You dont need to prove it).
  Can you apply the theorem to some special cases, say, where the calculations are simple.

- about the definition of $H^*(X)$ (cohomology)
  Starting from the singular chain complex $S_*(X)$ of X, how would you proceed to calculate the <u>cohomology groups</u> $H^k(X)$?

- For a given abelian group G, there is also the homology theory (singular) with G as <u>coefficient group</u>, namely we write

$$H_k(X) = H_k(X; \mathbb{Z}) \quad , \quad H_k(X; G)$$

Starting from the singular chain complex $S_*(X) = S_*(X; \mathbb{Z})$, as usual, how will you construct a chain complex from which one can calculate the groups $H_k(X; G)$?

[Here you need to know the "meaning" of the <u>tensor product</u> $G \otimes H \; (= G \otimes_{\mathbb{Z}} H)$ of two abelian groups G, H]

- How would you set up the <u>axioms</u> for a homology theory? (cf. page 74)