MA3402 HOMEWORK

Homeworks are intended as quick exercises to help solidify key ideas from the previous week. If you are spending a long time on them, stop and talk it over with someone.

1. For Thursday 28 September 2023

(1) Find all points $p \in \mathbb{R}^3$ such that $x, x^2 + y^2 + z^2 - 1, z$ form a coordinate system around p. Hint: you will need consequences of the Inverse Function Theorem.

(2) For $F\colon \mathbb{R}^n \to \mathbb{R}^m$ prove that the induced linear map $F_*\colon T_pN \to T_{F(p)}M$ is given by a Jacobian matrix.(The outline was covered in lectures.)

(3) Show that the differential of the identity map $\mathbb{I}_M\colon M \to M$ is also an identity map.

(4) For two overlapping charts $(U, x^1, \dots, x^n), (V, y^1, \dots, y^n)$ on a manifold M around a point p, show that

$$\frac{\partial}{\partial x^j} = \sum_i \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i}.$$

Hint: write out the bases for T_pU and T_pV.

(5) Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = x^2 + y^2 + z^2 - 1$. Show that 0 is a regular value of f. What are the critical points of f? What does $f^{-1}(0)$ look like?

2. Exercises

(1) Let $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ be given by $F(x, y, xy)$ for $(x, y) \in \mathbb{R}^2$. Let (u, v, w) be the standard coordinates in $\mathbb{R}^3$. For $p = (x, y) \in \mathbb{R}^2$, express $F_*(\frac{\partial}{\partial x}|_p)$ as a linear combination of $\frac{\partial}{\partial u}|_{F(p)}$, $\frac{\partial}{\partial v}|_{F(p)}$ and $\frac{\partial}{\partial w}|_{F(p)}$.

(2) Fix a real number α and define a map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x, y) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (u, v)$$

where x, y are the standard coordinates in the domain $\mathbb{R}^2$, u, v are the standard coordinates in the range $\mathbb{R}^2$. Let $X = -y\partial/\partial x + x\partial/\partial y$ be a vector field on $\mathbb{R}^2$. At $p = (x, y) \in \mathbb{R}^2$, show that $F_*(X_p) = (a\partial/\partial u + b\partial/\partial v)|_{F(p)}$ and find a and b in terms of x, y, α.

(3) Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x^3 - 6xy + y^2$. Find all values $c \in \mathbb{R}$ for which the preimage $f^{-1}(c)$ is a submanifold of $\mathbb{R}^2$.