EXERCISE SHEET 3

Exercise 1. Recall that the circle $S^1 \subset \mathbb{R}^2$ admits an atlas $\mathfrak{U} = \{(U_i^\epsilon, \varphi_i^\epsilon) \mid i \in \{0,1\}, \epsilon \in \{+,-\}\}$, where $U_i^+ = \{(r^0, r^1) \in \mathbb{R}^2 \mid r^i > 0\}$ and $U_i^- = \{(r^0, r^1) \in \mathbb{R}^2 \mid r^i < 0\}$ and where φ_i^+ and φ_0^- are given by projecting away from the ith coordinate. Is this an oriented atlas? If not, alter the coordinate functions φ_i^ϵ to make $\mathfrak{U}$ into an oriented atlas.

Solution. The atlas is not oriented. To see this, recall that an inverse to φ_1^+ is given by

$$(\varphi_1^+)^{-1} \colon (-1,1) \to U_1^+, \quad t \mapsto (t, \sqrt{1-t^2}).$$

Therefore, the transition map $\varphi_0^+ \circ (\varphi_1^+)^{-1}$ is given by

$$\varphi_0^+ \circ (\varphi_1^+)^{-1} \colon (0,1) \to (0,1), \quad t \mapsto \sqrt{1-t^2}.$$

Since the derivative of this map is given by $t \mapsto \frac{-t}{\sqrt{1-t^2}}$ which is negative everywhere on $(0,1)$, the claim follows. In order to obtain an oriented atlas, let us set $\psi_0^+ = -\varphi_0^+$, $\psi_1^+ = -\varphi_1^-$ and $\psi_i^\epsilon = \varphi_i^\epsilon$ for the remaining two charts. Then the atlas $\mathfrak{U}' = \{(U_i^\epsilon, \psi_i^\epsilon) \mid i \in \{0,1\}, \epsilon \in \{+,-\}\}$ is oriented. Intuitively, this is the case as all of the curves $(\varphi_i^\epsilon)^{-1} \colon (-1,1) \to U_i^\epsilon$ probe the circle in clockwise direction. Formally, the transition function $\psi_0^+ \circ (\psi_1^+)^{-1}$ is given by $t \mapsto -\sqrt{1-t^2}$ which has positive derivative everywhere. The case of the other transition functions is checked analogously. □

Exercise 2. Let M be a manifold with boundary ∂M. Cover ∂M by charts $(U_\alpha, x_\alpha^1, \dots, x_\alpha^n)_{\alpha \in A}$ in M. For each α, we define a vector field $X_\alpha = -\frac{\partial}{\partial x_\alpha^n}$ on $U_\alpha \cap \partial M$. Fix a partition of unity $(p_\alpha)_{\alpha \in A}$ subordinate to the cover $(U_\alpha \cap \partial M)_{\alpha \in A}$ of ∂M and define $X = \sum_\alpha p_\alpha X_\alpha$. Show that X is a smooth outward pointing vector field along ∂M.

Solution. Since for each α one has $X|_{U_\alpha} = -\frac{\partial}{\partial x_\alpha^n}$, the vector field X is smooth. In order to show that X is outward pointing, we will prove that $-X$ is inward-pointing. Since for each $p \in U_\alpha \cap \partial M$ the subspace $T_p(\partial M) \subset T_p(M)$ is spanned by $\frac{\partial}{\partial x_\alpha^1}, \dots, \frac{\partial}{\partial x_\alpha^{n-1}}$, we have $-X_p \notin T_p(\partial M)$. Let $r^1, \dots, r^n$ be the standard coordinates in $\mathbb{R}^n$, and let $\varphi_\alpha = (x_\alpha^1, \dots, x_\alpha^n)$. By choosing $\epsilon > 0$ such that the image of the function

$$\gamma \colon [0, \epsilon) \to \mathbb{H}^n, \quad t \mapsto \varphi_\alpha(p) + tr^n$$

is contained in $\varphi_\alpha(U_\alpha)$, we obtain a smooth curve $\delta = \varphi_\alpha^{-1} \circ \gamma \colon [0, \epsilon) \to U_\alpha$ that satisfies $\delta(0) = p$, $\delta((0,\epsilon)) \subset M^\circ$ and $\delta'(0) = -X_p$. By definition, this means that $-X_p$ is inward pointing. □

Exercise 3. Let M be an oriented manifold with boundary ∂M, let ω be an orientation form on M and X be an outward pointing smooth vector field along ∂M.

(1) If τ is another orientation form on M, then $\tau = f\omega$ for some $f \in C^\infty(M)$ satisfying $f > 0$ everywhere on M. Show that $\iota_X \tau = f \iota_X \omega$.
(2) Show that if Y is another ourward pointing smooth vector field along ∂M, there is $g \in C^\infty(\partial M)$ satisfying $g > 0$ everywhere on ∂M such that $\iota_X \omega = g \iota_Y \omega$.

Solution. Suppose that the dimension of M is n. Given $p \in \partial M$, let $X_1, \dots, X_{n-1}$ be arbitrary vectors in $T_p(\partial M)$.

Date: November 16, 2021.

(1) We compute

$$\iota_X\tau_p(X_1,\dots,X_{n-1}) = \tau_p(X,X_1,\dots,X_{n-1}) = f(p)\omega_p(X,X_1,\dots,X_{n-1}) = f(p)\iota_X\omega_p(X_1,\dots,X_{n-1}).$$

Hence $\iota_X\tau = f\iota_X\omega$.

(2) Cover ∂M with charts $(U_\alpha, x^1_\alpha,\dots,x^n_\alpha)$ on M, and pick a partition of unity (p_α) subordinate to $(U_\alpha\cap\partial M)$. Then $Z=\sum_\alpha -\frac{\partial}{\partial x^n}$ is an outward pointing smooth vector field along ∂M. Suppose that there are $h,k\in C^\infty(\partial M)$ with $h>0$ and $k>0$ everywhere on ∂M such that $\iota_X\omega = h\iota_Z\omega$ and $\iota_Y\omega = k\iota_Z\omega$. Then $g=h/k\in C^\infty(\partial M)$ is well-defined and positive everywhere on ∂M and furthermore gives rise to the desired identity $\iota_X\omega = g\iota_Y\omega$. As a consequence of this observation, we may assume without loss of generality that $Y=Z$.

For each α and each $p\in U_\alpha\cap\partial M$, there are $a^1_\alpha(p),\dots,a^n_\alpha(p)$ such that

$$X_p = \sum_{i=1}^n a^i_\alpha(p)\frac{\partial}{\partial x^i_\alpha}|_p.$$

Note that since X_p is outward pointing, we must have $a^\alpha_p<0$. Using this expansion, we may now compute

$$\begin{aligned}\iota_X\omega_p(\frac{\partial}{\partial x^1_\alpha}|_p,\dots,\frac{\partial}{\partial x^{n-1}_\alpha}|_p) &= \omega_p(X,\frac{\partial}{\partial x^1_\alpha}|_p,\dots,\frac{\partial}{\partial x^{n-1}_\alpha}|_p)\\ &= \sum_{i=1}^n a^i_\alpha(p)\omega_p(\frac{\partial}{\partial x^i_\alpha}|_p,\frac{\partial}{\partial x^1_\alpha}|_p,\dots,\frac{\partial}{\partial x^{n-1}_\alpha}|_p)\\ &= -a^n_\alpha(p)\omega_p(-\frac{\partial}{\partial x^n}|_p,\frac{\partial}{\partial x^1_\alpha}|_p,\dots,\frac{\partial}{\partial x^{n-1}_\alpha}|_p)\\ &= -a^n_\alpha(p)\iota_Z\omega(\frac{\partial}{\partial x^1_\alpha}|_p,\dots,\frac{\partial}{\partial x^{n-1}_\alpha}|_p).\end{aligned}$$

Note that as X is smooth, the function $a^n_\alpha\colon U_\alpha\cap\partial M\to\mathbb{R}$ is smooth as well. Hence, the function $g=\sum_\alpha -p_\alpha a^n_\alpha\in C^\infty(\partial M)$ is positive everywhere on M and satisfies the desired identity $\iota_X\omega = g\iota_Z\omega$. □

Exercise 4. Let M be the left half space $\{(x^1,\dots,x^n)\in\mathbb{R}^n \mid x^1\le 0\}$ with orientation form $\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n$. Show that an orientation form for the boundary orientation on $\partial M = \{(0,x^2,\dots,x^n)\in\mathbb{R}^n\}\simeq\mathbb{R}^{n-1}$ is $\mathrm{d}x^2\wedge\cdots\wedge\mathrm{d}x^n$.

Solution. At each point $p\in\partial M$, the vectors $\frac{\partial}{\partial x^2},\dots,\frac{\partial}{\partial x^n}$ define a basis for $T_p(\partial M)$. Therefore, every $(n-1)$-form on ∂M is uniquely determined by its evaluation at $\frac{\partial}{\partial x^2},\dots,\frac{\partial}{\partial x^n}$. On account of the identity $\mathrm{d}x^2\wedge\cdots\wedge\mathrm{d}x^n(\frac{\partial}{\partial x^2},\dots,\frac{\partial}{\partial x^n})=1$, it therefore suffices to construct an orientation form for the boundary orientation on ∂M that satisfies the same identity. Now the vector field $X=\frac{\partial}{\partial x^1}$ clearly defines an outward pointing smooth vector field along ∂M. Therefore, the contraction of the orientation form $\mathrm{d}^1\wedge\cdots\wedge\mathrm{d}x^n$ on M along this vector field yields

$$\iota_X\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n(\frac{\partial}{\partial x^2},\dots,\frac{\partial}{\partial x^n}) = \mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n(\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n})=1,$$

hence the claim follows. □

Exercise 5. Orient the unit sphere $S^n\subset\mathbb{R}^{n+1}$ as the boundary of the closed unit ball. Let $U=\{(x^0,\dots,x^n)\in S^n \mid x^n>0\}$ be the upper unit hemisphere. Recall that $(U,x^0,\dots,x^{n-1})$ is a chart for S^n.

(1) Show that an orientation form on S^n is given by

$$\omega = \sum_{i=0}^{n} (-1)^i x^i \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \cdots \wedge \mathrm{d}x^n.$$

(2) Find an orientation form on U in terms of $\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}$.

(3) Show that the projection map $(x^0, \ldots, x^{n-1})\colon U \to \mathbb{R}^n$ is orientation-preserving if and only if n is even.

Solution.

(1) Let D^{n+1} be the closed unit ball in $\mathbb{R}^{n+1}$. The vector field $X = \sum_{i=0}^{n} x^i \frac{\partial}{\partial x^i}$ on D^{n+1} is an outward pointing vector field along S^n. Let $\omega = \iota_X \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n$. Fix a point $p \in S^n$. Using a result from lectures for the contraction of a product of n 1-covectors (proposition 20.7 in the book), we obtain

$$\begin{aligned}\omega_p &= \sum_{i=0}^{n} (-1)^i \mathrm{d}x^i(X_p) \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \cdots \wedge \mathrm{d}x^n \\ &= \sum_{i=0}^{n} (-1)^i x^i(p) \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \cdots \wedge \mathrm{d}x^n\end{aligned}$$

where the second equality follows from

$$\mathrm{d}x^i(X_p) = \sum_{j=0}^{n} x^j(p) \mathrm{d}x^i\left(\frac{\partial}{\partial x^j}|_p\right) = x^i(p).$$

As $p \in S^n$ was chosen arbitrarily, the claim follows.

(2) Let $M = D^{n+1} \cap \{(x^0, \ldots, x^n) \in \mathbb{R}^{n+1} \mid x^n > 0\}$. Then $\partial M = U$. The vector field $X = \frac{\partial}{\partial x^n}$ on M is outward pointing along U. Therefore, the orientation form $\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n$ on M gives rise to an orientation form $\omega = \iota_X \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n$. Given $p \in U$, proposition 20.7 in the book again implies

$$\omega_p = \sum_{i=0}^{n} (-1)^i \mathrm{d}x^i(X_p) \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \cdots \wedge \mathrm{d}x^n.$$

Since $\mathrm{d}x^i(X_p) = \mathrm{d}x^i(\frac{\partial}{\partial x^n}) = \delta_{in}$ (where δ_{in} is the Kronecker delta), we conclude

$$\omega = (-1)^n \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}.$$

(3) The projection map $(x^0, \ldots, x^{n-1})$ is orientation-preserving if and only if the pullback of the orientation form $\mathrm{d}x^1 \wedge \cdots \mathrm{d}r^n$ on $\mathbb{R}^n$ along $(x^0, \ldots, x^{n-1})$ is equivalent to the orientation form $\omega = (-1)^n \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}$ from the previous part of the exercise. Since one has

$$(x^0, \ldots, x^{n-1})^* \mathrm{d}r^1 \wedge \cdots \wedge \mathrm{r}^n = \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1},$$

this is the case if and only if $(-1)^n = 1$, i.e. if and only if n is even. □

Exercise 6. Let us fix two real numbers $R > r > 0$. Then

$$\mathbb{T}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid (\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2\} \subset \mathbb{R}^3$$

is the torus in which the tube has radius r and the distance from the centre of the tube to the centre of the torus is equal to R. Moreover,

$$\mathbb{T}^2_S = \{(x, y, z) \in \mathbb{R}^3 \mid (\sqrt{x^2 + y^2} - R)^2 + z^2 \leq r^2\} \subset \mathbb{R}^3$$

is the associated *solid* torus in which the interior of the tube is included. This is a manifold with boundary $\partial\mathbb{T}^2_S = \mathbb{T}^2$ that inherits the orientation of the ambient manifold $\mathbb{R}^3$. Let us equip $\mathbb{T}^2$ with the induced orientation of $\mathbb{T}^2_S$. Compute the integral

$$\int_{\mathbb{T}^2} z\mathrm{d}x \wedge \mathrm{d}y.$$

Solution. Using Stokes' theorem, one has

$$\int_{\mathbb{T}^2} z\mathrm{d}x \wedge \mathrm{d}y = \int_{\mathbb{T}^2_S} \mathrm{d}(z\mathrm{d}x \wedge \mathrm{d}y) = \int_{\mathbb{T}^2_S} \mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y = \int_{\mathbb{T}^2_S} \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

Therefore, the integral is equal to the volume of $\mathbb{T}^2_S$. Hence

$$\int_{\mathbb{T}^2} z\mathrm{d}x \wedge \mathrm{d}y = 2\pi^2 r^2 R.$$ □

Exercise 7. Let M be an orientable compact manifold with boundary ∂M. Prove that the inclusion $i\colon \partial M \hookrightarrow M$ does not admit a smooth retraction, i.e. a smooth map $r\colon M \to \partial M$ satisfying $ri = \mathrm{id}_{\partial M}$. **Hint**: Assume that there exists such a retraction r. Derive a contradiction by considering an orientation form on ∂M and by using Stokes' theorem.

Solution. Note that since M is compact, every n-form on M are integrable. Similarly, every $(n-1)$-form on ∂M is integrable. Now suppose that there is a smooth retraction r. Choose an orientation form ω on ∂M. Since ω is an $(n-1)$-form (where n is the dimension of M), we have $\mathrm{d}\omega = 0$. By Stokes' theorem, this implies

$$0 = \int_M r^*\mathrm{d}\omega = \int_M \mathrm{d}r^*\omega = \int_{\partial M} r^*\omega.$$

On the other hand, since r is a retraction of i, one has $i^*r^*\omega = \omega$. Therefore

$$\int_{\partial M} r^*\omega = \int_{\partial M} \omega.$$

But since ω is an orientation form and therefore everywhere positive, one must have

$$\int_{\partial M} \omega > 0,$$

which is a contradiction. □