EXERCISE SHEET 3

Exercise 1. Recall that the circle $S^1 \subset \mathbb{R}^2$ admits an atlas $\mathfrak{U} = \{(U_i^\epsilon, \varphi_i^\epsilon) \mid i \in \{0,1\}, \epsilon \in \{+,-\}\}$, where $U_i^+ = \{(r^0, r^1) \in \mathbb{R}^2 \mid r^i > 0\}$ and $U_i^- = \{(r^0, r^1) \in \mathbb{R}^2 \mid r^i < 0\}$ and where φ_i^+ and φ_0^- are given by projecting away from the ith coordinate. Is this an oriented atlas? If not, alter the coordinate functions φ_i^ϵ to make $\mathfrak{U}$ into an oriented atlas.

Exercise 2. Let M be a manifold with boundary ∂M. Cover ∂M by charts $(U_\alpha, x_\alpha^1, \dots, x_\alpha^n)_{\alpha \in A}$ in M. For each α, we define a vector field $X_\alpha = -\frac{\partial}{\partial x_\alpha^n}$ on $U_\alpha \cap \partial M$. Fix a partition of unity $(p_\alpha)_{\alpha \in A}$ subordinate to the cover $(U_\alpha \cap \partial M)_{\alpha \in A}$ of ∂M and define $X = \sum_\alpha p_\alpha X_\alpha$. Show that X is a smooth outward pointing vector field along ∂M.

Exercise 3. Let M be an oriented manifold with boundary ∂M, let ω be an orientation form on M and X be an outward pointing smooth vector field along ∂M.

(1) If τ is another orientation form on M, then $\tau = f\omega$ for some $f \in C^\infty(M)$ satisfying $f > 0$ everywhere on M. Show that $\iota_X \tau = f \iota_X \omega$.

(2) Show that if Y is another ourward pointing smooth vector field along ∂M, there is $g \in C^\infty(\partial M)$ satisfying $g > 0$ everywhere on ∂M such that $\iota_X \omega = g \iota_Y \omega$.

Exercise 4. Let M be the left half space $\{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^1 \leq 0\}$ with orientation form $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. Show that an orientation form for the boundary orientation on $\partial M = \{(0, x^2, \dots, x^n) \in \mathbb{R}^n\} \simeq \mathbb{R}^{n-1}$ is $\mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n$.

Exercise 5. Orient the unit sphere $S^n \subset \mathbb{R}^{n+1}$ as the boundary of the closed unit ball. Let $U = \{(x^0, \dots, x^n) \in S^n \mid x^n > 0\}$ be the upper unit hemisphere. Recall that $(U, x^0, \dots, x^{n-1})$ is a chart for S^n.

(1) Show that an orientation form on S^n is given by

$$\omega = \sum_{i=0}^{n} (-1)^i x^i \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \cdots \wedge \mathrm{d}x^n.$$

(2) Find an orientation form on U in terms of $\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{n-1}$.

(3) Show that the projection map $(x^0, \dots, x^{n-1}) \colon U \to \mathbb{R}^n$ is orientation-preserving if and only if n is even.

Exercise 6. Let us fix two real numbers $R > r > 0$. Then

$$\mathbb{T}^2 = \{(x,y,z) \in \mathbb{R}^3 \mid (\sqrt{x^2+y^2} - R)^2 + z^2 = r^2\} \subset \mathbb{R}^3$$

is the torus in which the tube has radius r and the distance from the centre of the tube to the centre of the torus is equal to R. Moreover,

$$\mathbb{T}_S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid (\sqrt{x^2+y^2} - R)^2 + z^2 \leq r^2\} \subset \mathbb{R}^3$$

is the associated *solid* torus in which the interior of the tube is included. This is a manifold with boundary $\partial \mathbb{T}_S^2 = \mathbb{T}^2$ that inherits the orientation of the ambient manifold $\mathbb{R}^3$. Let us equip $\mathbb{T}^2$ with the induced

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orientation of $\mathbb{T}^2_S$. Compute the integral

$$\int_{\mathbb{T}^2} z \mathrm{d}x \wedge \mathrm{d}y.$$

Exercise 7. Let M be an orientable compact manifold with boundary ∂M. Prove that the inclusion $i\colon \partial M \hookrightarrow M$ does not admit a smooth retraction, i.e. a smooth map $r\colon M \to \partial M$ satisfying $ri = \mathrm{id}_{\partial M}$. **Hint**: Assume that there exists such a retraction r. Derive a contradiction by considering an orientation form on ∂M and by using Stokes' theorem.