EXERCISE SHEET 2

***Exercise 1.** Let M be a smooth manifold and let $p \in M$ be a point. The goal of this exercise is to compare different definitions for the tangent space T_pM of M at p.

Let $\mathrm{Path}_p(M)$ be the set of smooth functions $\gamma\colon \mathbb{R} \to M$ that satisfy $\gamma(0) = p$. Recall that we define the stalk $C_p^\infty(M)$ as the set of equivalence classes of smooth maps $f\colon U \to \mathbb{R}$ where $U \subset M$ is an open neighbourhood of p and where two maps $f\colon U \to \mathbb{R}$ and $g\colon V \to \mathbb{R}$ are equivalent if they coincide on some open neighbourhood $W \subset U \cap V$ of p.

(1) Define

$$D_\gamma(f) = \frac{\mathrm{d}(f \circ \gamma)}{\mathrm{d}t}(0),$$

where $\gamma \in \mathrm{Path}_p(M)$ and where $f\colon U \to \mathbb{R}$ is a smooth map that is defined on an open neighbourhood $U \subset M$ of p. Show that D_γ determines a well-defined derivation

$$D_\gamma\colon C_p^\infty(M) \to \mathbb{R}.$$

Conclude that one obtains a map

$$D\colon \mathrm{Path}_p(M) \to T_p(M), \quad \gamma \mapsto D_\gamma.$$

(2) Show that the map $D\colon \mathrm{Path}(M) \to T_p(M)$ is surjective.

(3) Conclude that the map D descends to a bijection

$$\mathrm{Path}_p(M)_{/\sim} \simeq T_p(M)$$

where $\gamma \sim \delta$ if and only if there is a chart (U, φ) around p such that $\frac{\mathrm{d}(\varphi\circ\gamma)}{\mathrm{d}t}(0) = \frac{\mathrm{d}(\varphi\circ\delta)}{\mathrm{d}t}(0)$.

As a consequence of part (3), we may regard a tangent vector at $p \in M$ as an equivalence class of smooth paths passing through p. The correspondence between such paths and tangent vectors admits a geometric interpretation: locally, i.e. when M is an open subset of $\mathbb{R}^n$, every $\gamma \in \mathrm{Path}_p(M)$ is equivalent to exactly one straight line that passes through p, the line given by

$$\mathbb{R} \ni t \mapsto p + t\frac{\mathrm{d}\gamma}{\mathrm{d}t}(0).$$

The equivalence from (3) carries this line to the derivation that is given by the directional derivative along the vector $\frac{\mathrm{d}\gamma}{\mathrm{d}t}(0) \in \mathbb{R}^n$.

Recall that we denote by $C^\infty(M)$ the vector space of smooth functions $M \to \mathbb{R}$, where addition and scalar multiplication are defined pointwise. Note that this is a *ring* (in fact, an $\mathbb{R}$-algebra) when defining multiplication pointwise as well. Let $I_p \subset C^\infty(M)$ be the subspace of functions vanishing at p, i.e. those functions $f\colon M \to \mathbb{R}$ that satisfy $f(p) = 0$. This is an *ideal* in $C^\infty(M)$. Define $I_p^2 \subset I_p$ as the subspace generated by the set $\{fg \mid f, g \in I_p\}$. Explicitly, I_p^2 is comprised of those smooth functions $f\colon M \to \mathbb{R}$ that can be written as a linear combination

$$f = \sum_{i=1}^{n} f_i g_i$$

with $f_i, g_i \in I_p$ for all $i = 1, \ldots, n$.

Date: October 23, 2021.

(4) Construct an isomorphism of vector spaces $T_p(M) \simeq (I_p/I_p^2)^\vee$. **Hint**: You may use without proof the smooth version of *Urysohn's lemma*: For any $K \subset U \subset M$ where K is closed and U is open, there is a smooth function $m\colon M \to \mathbb{R}$ satisfying $m|_K = 1$ and $M|_{M\setminus U} = 0$.

The isomorphism in part (4) gives rise to an identification of the cotangent space $T_p^\vee(M)$ with I_p/I_p^2. The geometric intuition behind this is as follows: if $f\colon M \to \mathbb{R}$ is a smooth function, Taylor's theorem implies that locally around p this function is given by a Taylor series

$$f(q) = f(p) + \sum_i \frac{\partial f}{\partial x^i}(p)(q^i - p^i) + o(|q-p|^2).$$

By applying the affine coordinate transformation $q \mapsto q + p$, we may assume without loss of generality $p = 0$. Now $f \in I_p$ precisely if the first term vanishes, and $f \in I_p^2$ precisely if the first two terms vanish. Consequently, every class $[f] \in I_p/I_p^2$ has a unique representative that is given by the map

$$\mathbb{R}^n \ni q \mapsto \sum_i \frac{\partial f}{\partial x^i}(p) q^i \in \mathbb{R},$$

i.e. by the covector $(\frac{\partial f}{\partial x^1}(p), \dots, \frac{\partial f}{\partial x^n}(p)) \in T_p^\vee(M)$.

Solution.

(1) The map D_γ is well-defined: given two functions $f\colon U \to \mathbb{R}$ and $f'\colon V \to \mathbb{R}$ that represent the same object in the stalk $C_p^\infty(M)$, there is an open neighbourhood $W \subset U \cap V$ around p such that $f|_W = f'|_W$, hence $f \circ \gamma|_{\gamma^{-1}(W)} = f' \circ \gamma|_{\gamma^{-1}(W)}$. Since $\gamma^{-1}(W)$ is an open neighbourhood around 0, one must have $D_\gamma(f) = D_\gamma(f')$. Now given $[f], [g] \in C_p^\infty(M)$, we can find an open neighbourhood $U \subset M$ around p and representatives $f\colon U \to \mathbb{R}$ and $g\colon U \to \mathbb{R}$ of $[f]$ and $[g]$. By definition of addition in $C_p^\infty(M)$, one has $[f] + [g] = [f+g]$, and since $(f+g)\circ\gamma = f\circ\gamma + g\circ\gamma$, linearity of the derivative implies $D_\gamma(f+g) = D_\gamma(f) + D_\gamma(g)$. A similar argument shows that D_γ commutes with scalar multiplication. Hence D_γ is $\mathbb{R}$-linear. Moreover, the identity $(fg)\circ\gamma = (f\circ\gamma)(g\circ\gamma)$ shows together with the product rule of the derivative that D_γ is a derivation. Hence D_γ is a well-defined element of $T_p(M)$, which shows that one gets the desired map $D\colon \mathrm{Path}_p(M) \to T_p(M)$.

(2) Choose a chart $(U, x^1, \dots, x^n)$ around p. We may assume without loss of generality $x^i(p) = 0$ for all $i = 1, \dots, n$. As the set $\{\frac{\partial}{\partial x^1}|_p, \dots, \frac{\partial}{\partial x^n}|_p\}$ is a basis for $T_p(M)$, we may write

$$D = \sum_i \lambda_i \frac{\partial}{\partial x^i}|_p$$

for any derivation $D \in T_p(M)$. Let

$$l\colon \mathbb{R} \to \mathbb{R}^n, \quad t \mapsto t(\lambda_1, \dots, \lambda_n)$$

be the line spanned by $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, and let $\gamma = \varphi^{-1} \circ l$ be the induced path in M. Let $f\colon U \to \mathbb{R}$ be a smooth function, and let us set $g = f \circ \varphi^{-1}\colon \mathbb{R}^n \to \mathbb{R}$. We now compute

$$D_\gamma(f) = \frac{\mathrm{d}(f\circ\gamma)}{\mathrm{d}t}(0) = \frac{\mathrm{d}(g \circ l)}{\mathrm{d}t}(0) = \sum_i \lambda_i \frac{\partial g}{\partial r^i}(0) = \sum_i \lambda_i \frac{\partial f}{\partial x^i}(p) = D_\gamma(f),$$

where $r^1, \dots, r^n$ denote the standard coordinates of $\mathbb{R}^n$. Hence D is surjective.

(3) Given $\gamma, \delta \in \mathrm{Path}_p(M)$, one has $D_\gamma = D_\delta$ if and only if for every open neighbourhood U around p and every smooth function $f\colon U \to \mathbb{R}$, one has

$$\tfrac{\mathrm{d}(f\circ\gamma)}{\mathrm{d}t}(0) = \tfrac{\mathrm{d}(f\circ\delta)}{\mathrm{d}t}(0).$$

Now suppose that $D_\gamma = D_\delta$, and choose a chart $(U, x^1, \dots, x^n)$ around p. The fact that $x^i \colon U \to \mathbb{R}$ is smooth implies $\frac{\mathrm{d}(x^i\circ\gamma)}{\mathrm{d}t}(0) = \frac{\mathrm{d}(x^i\circ\delta)}{\mathrm{d}t}(0)$ for all i. In other words, setting $\varphi = (x^1, \dots, x^n)$, one finds $\frac{\mathrm{d}(\varphi\circ\gamma)}{\mathrm{d}t}(0) = \frac{\mathrm{d}(\varphi\circ\delta)}{\mathrm{d}t}(0)$.

Conversely, suppose that there is a chart $(U, x^1, \dots, x^n)$ around p such that $\frac{\mathrm{d}(\varphi\circ\gamma)}{\mathrm{d}t}(0) = \frac{\mathrm{d}(\varphi\circ\delta)}{\mathrm{d}t}(0)$, where we again set $\varphi = (x^1, \dots, x^n)$. If f is a smooth function defined in a neighbourhood of p, shrinking U if necessary implies that we may assume without loss of generality that f is defined on U. Setting $g = \varphi^{-1} \circ f$, the chain rule implies

$$\frac{\mathrm{d}(f\circ\gamma)}{\mathrm{d}t}(0) = \frac{\mathrm{d}(g\circ\varphi\circ\gamma)}{\mathrm{d}t}(0) = \sum_i \frac{\partial g}{\partial r^i}(\varphi(p))\frac{\mathrm{d}(x^i\circ\gamma)}{\mathrm{d}t}(0),$$

hence $\frac{\mathrm{d}(f\circ\gamma)}{\mathrm{d}t}(0) = \frac{\mathrm{d}(f\circ\delta)}{\mathrm{d}t}(0)$.

(4) To begin with, note that assigning to a smooth function $f \in I_p$ its germ $[f] \in C_p^\infty(M)$ defines a linear map $[-]\colon I_p \to C_p^\infty(M)$. Therefore, precomposition with $[-]$ gives rise to a linear map

$$[-]^*\colon C^\infty(M)^\vee \to I_p^\vee.$$

Since every derivation $D \in T_p(M)$ is by definition a linear map $C_p^\infty(M) \to \mathbb{R}$, we have an inclusion of vector spaces $T_p(M) \subset C_p^\infty(M)^\vee$. Restricting $[-]^*$ to this subspace therefore gives rise to a well-defined linear map

$$[-]^*\colon T_p(M) \to I_p^\vee.$$

Furthermore, note that precomposition with the quotient map $\pi\colon I_p \twoheadrightarrow I_p/I_p^2$ induces an injective map

$$\pi^*\colon (I_p/I_p^2)^\vee \hookrightarrow I_p^\vee$$

whose image is given by those linear maps $f\colon I_p \to \mathbb{R}$ that satisfy $f(I_p^2) = 0$. Given $f, g \in I_p$ and a derivation $D \in T_p(M)$, the identity

$$D([fg]) = f(p)D([g]) + D([f])g(p)$$

implies $D([fg]) = 0$ since by assumption $f(p) = 0 = g(p)$. Hence $D(I_p^2) = 0$, which implies that the linear map $[-]^*$ factors through the inclusion $(I_p/I_p^2)^\vee \hookrightarrow I_p^\vee$ and therefore defines a linear map

$$[-]^*\colon T_p(M) \to (I_p/I_p^2)^\vee.$$

Now suppose that $D \in T_p(M)$ such that $[-]^*(D) = D \circ [-] = 0$. This means that $D([f]) = 0$ for all $f \in I_p$. Let $[g] \in C_p^\infty(M)$ be an arbitrary germ, and let $g'\colon U \to \mathbb{R}$ be an arbitrary representative of $[g]$ defined on some open neighbourhood U of p. Using Urysohn's lemma, we may choose open neighbourhoods $W \subsetneq V \subsetneq U$ of p and a smooth function $m\colon M \to \mathbb{R}$ satisfying $m|_{\overline{W}} = 1$ and $m|_{M\setminus V} = 0$. Then the function

$$g\colon M \to \mathbb{R}, \quad q \mapsto \begin{cases} g'(q)m(q), & q \in U, \\ m(q), & q \notin U \end{cases}$$

is smooth and represents the same germ $[g] \in C_p^\infty(M)$. In other words, given $[g] \in C_p^\infty(M)$ we can always find a *globally* defined representative $g \in C^\infty(M)$ of $[g]$. Let $1_M \in C^\infty(M)$ be the function defined by $1_M(q) = 1$ for all $q \in M$. Since $1_M 1_M = 1_M$ and therefore $D([1_M]) = 2D([1_M])$, one must always have $D([1_M]) = 0$. As a consequence, the function $f = g - g(p)1_M$ satisfies $f(p) = 0$, i.e. defines an element of I_p. By assumption, $D([f]) = 0$, and we conclude

$$0 = D([f]) = D([g]) - g(p)D([1_M]) = D([g]).$$

Since $[g]$ was chosen arbitrarily, we conclude that $D = 0$. Therefore the map $[-]^*$ is injective. To show that it is surjective, let $\varphi\colon I_p/I_p^2 \to \mathbb{R}$ be a linear map. Given a germ $[f] \in C_p^\infty(M)$, choose a representative $f \in C^\infty(M)$ as above and set

$$D_\varphi([f]) = \varphi(f - 1_M f(p)).$$

Let us first show that $D_\varphi([f])$ does not depend on the chosen representative for $[f]$. To that end, note that given $g \in C^\infty(M)$ such that $[g] = 0$, there is an open neighbourhood U of p such that $g|_U = 0$. Choose open neighbourhoods $W \subsetneq V \subsetneq U$ of p. By applying Urysohn's lemma to $M \setminus V$ and $M \setminus \overline{W}$, there is a smooth function $m\colon M \to \mathbb{R}$ satisfying $m|_W = 0$ and $m|_{M\setminus V} = 1$. One therefore has $gm = g$. Therefore, if $f' \in C^\infty(M)$ is another representative for $[f]$, there exists a smooth function $m \in C^\infty(M)$ such that $f - f' = (f - f')m$. We therefore compute

$$\varphi(f - 1_M f(p)) - \varphi(f' - 1_M f'(p)) = \varphi(f - f') = \varphi((f - f')m) = 0$$

where the rightmost equality follows from $(f - f')m \in I_p^2$. We thus conclude that $D_\varphi([f])$ does not depend on the chosen representative and therefore gives rise to a well-defined map $D_\varphi\colon T_p(M) \to \mathbb{R}$. Since φ is linear, so is D_φ. We next show that D_φ is a derivation. Given $f, g \in C^\infty(M)$, we compute

$$\begin{aligned}
0 &= \varphi((f - 1_M f(p))(g - 1_M g(p))) \\
&= \varphi(fg - fg(p) - f(p)g + 1_M f(p)g(p)) \\
&= \varphi(fg - 1_M f(p)g(p)) - \varphi(fg(p) + f(p)g - 1_M 2f(p)g(p)) \\
&= \varphi(fg - 1_M f(p)g(p)) - f(p)\varphi(g - 1_M g(p)) - \varphi(f - 1_M f(p))g(p) \\
&= D_\varphi([fg]) - f(p)D_\varphi([g]) - D_\varphi([f])g(p),
\end{aligned}$$

where the first equality follows from $(f - 1_M f(p))(g - 1_M g(p)) \in I_p^2$. Thus $D_\varphi \in T_p(M)$. It is clear from the construction that $[-]^*(D_\varphi) = \varphi$, hence the map $[-]^*$ is surjective.

□

Exercise 2. Let $X_1, \ldots, X_n$ be n vector fields on an n-dimensional manifold M. Suppose that at $p \in M$, the vectors $(X_1)_p, \ldots, (X_n)_p$ are linearly independent. Show that there is a chart $(V, x^1, \ldots, x^n)$ around p such that $(X_i)_p = (\frac{\partial}{\partial x^i})_p$ for all $i = 1, \ldots, n$.

Solution.

Method 1: Let (U, φ) be an arbitrary chart around p, and let us write $\varphi = (y^1, \ldots, y^n)$ in coordinates. Let $L\colon \mathbb{R}^n \simeq T_p(M)$ be the isomorphism of vector spaces that carries the canonical basis vector $e^i \in \mathbb{R}^n$ to $\frac{\partial}{\partial y^i}|_p \in T_p(M)$. Since the vectors $(X_1)_p, \ldots, (X_n)_p$ are by assumption linearly independent and therefore form a basis of $T_p(M)$, the linear map defined by

$$f\colon T_p(M) \to T_p(M), \quad (X_i)_p \mapsto \frac{\partial}{\partial y^i}|_p$$

is an isomorphism. We therefore obtain an isomorphism $g = L^{-1} f L\colon \mathbb{R}^n \simeq \mathbb{R}^n$ of vector spaces. Let $(g_{ij})_{1 \le i,j \le n}$ be the associated coefficient matrix. Recall that any linear map is in particular smooth. We may therefore regard g as a diffeomorphism of smooth manifolds. When interpreted as such, the pushforward g_* is given by

$$g_*\colon T_q(\mathbb{R}^n) \to T_{g(q)}(\mathbb{R}^n), \quad \frac{\partial}{\partial r^i}|_q \mapsto \sum_k g_{ik} \frac{\partial}{\partial r^k}|_{g(q)}$$

for every $q \in \mathbb{R}^n$. In other words, if $M\colon \mathbb{R}^n \simeq T_q(\mathbb{R}^n)$ is the isomorphism that carries $e^i \in \mathbb{R}^n$ to $\frac{\partial}{\partial r^i}|_q$ and if $N\colon \mathbb{R}^n \simeq T_g(q)(\mathbb{R}^n)$ is the isomorphism that takes e^i to $\frac{\partial}{\partial r^i}|_{g(q)}$, we find $g_* = NgM^{-1}$.

As a consequence, $g_* = NL^{-1}fLM^{-1}$. Note that by construction the linear map ML^{-1} carries $\frac{\partial}{\partial y^i}|_p$ to $\frac{\partial}{\partial r^i}|_p$, hence we must have $ML^{-1} = \varphi_*$. Now the pair $(U, g \circ \varphi)$ defines a chart of M around p. Let $x^1, \dots, x^n$ be the coordinates of φ. By using the chain rule, we compute

$$(g\varphi)_*(X_i)_p = g_*\varphi_*(X_i)_p = NL^{-1}fLM^{-1}ML^{-1}(X_i)_p = NL^{-1}f(X_i)_p = NL^{-1}\frac{\partial}{\partial y^i}|_p = \frac{\partial}{\partial r^i}|_{g\varphi(p)},$$

from which we deduce that $(X_i)_p = \frac{\partial}{\partial x^i}|_p$ holds.

Method 2: Let $(U, y^1, \dots, y^n)$ be an arbitrary chart around p. Then we can write $(X_i)_p$ in terms of the corresponding basis of $T_p(U)$

$$(X_i)_p = \sum_k a_{ik}\frac{\partial}{\partial y^k}|_p, \quad a_{ik} \in \mathbb{R}.$$

We would like to define a coordinate system $x^1, \dots, x^n$ so that $(X_i)_p = \frac{\partial}{\partial x^i}|_p$. Suppose we had such a coordinate system, so

$$\frac{\partial}{\partial x^i}|_p = \sum_k a_{ik}\frac{\partial}{\partial y^k}|_p.$$

By applying both sides to a coordinate map y^j, we see that

$$\frac{\partial y^j}{\partial x^i}|_p = a_{ij}.$$

This is true if $y^j = \sum_i a_{ij}x^i$. Let $A = (a_{ij})$ be the matrix of the coefficients a_{ij}. Then we rewrite this expression as

$$\begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix} = A \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}.$$

Since $(X_j)_p$ are linearly independent, the matrix A is invertible

$$\begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} = A^{-1} \begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix}.$$

Hence we define $x^i = \sum_j (A^{-1})_{ij}y^j$ on U to be suitable coordinate charts such that $(X_i)_p = \frac{\partial}{\partial x^i}|_p$.

□

***Exercise 3.** Let $S^n = \{(x^0, \dots, x^n) \in \mathbb{R}^{n+1} \mid \sum_i (x^i)^2 = 1\} \subset \mathbb{R}^{n+1}$ be the unit sphere. We equip S^n with the subspace topology of $\mathbb{R}^{n+1}$. For $i = 0, \dots, n+1$, define subsets

$$U_i^+ = \{(x^0, \dots, x^n) \in S^n \mid x^i > 0\} \subset S^n$$

and

$$U_i^- = \{(x^0, \dots, x^n) \in S^n \mid x^i < 0\} \subset S^n.$$

(1) Show that $U_i^+ \subset S^n$ and $U_i^- \subset S^n$ is open for every $i = 0, \dots, n$. **Hint**: Realise both U_i^+ and U_i^- as a preimage of an open interval in $\mathbb{R}^n$ along the coordinate projection $x^i \colon S^n \subset \mathbb{R}^{n+1} \to \mathbb{R}$.

(2) Show that $\bigcup_{i=0}^n (U_i^+ \cup U_i^-) = S^n$.

For every $i = 0, \dots n$, let

$$\varphi_i \colon S^n \to \mathbb{R}^n, \quad (x^0, \dots, x^n) \mapsto (x^0, \dots, x^{i-1}, x^{i+1}, \dots, x^n)$$

be the projection away from the ith coordinate, i.e. the projection onto the coordinate hyperplane that is perpendicular to the ith coordinate axis. Observe that $\varphi_i(U_i^+) = \varphi_i(U_i^-) = D^n$, where

$$D^n = \{(y^1, \ldots, y^n) \in \mathbb{R}^n \mid \sum_j (y^j)^2 < 1\}$$

is the open unit disk in $\mathbb{R}^n$. By restricting φ_i to U_i^+ and U_i^-, we therefore obtain maps $\varphi_i^+ : U_i^+ \to D^n$ and $\varphi_i^- : U_i^- \to D^n$.

(3) Show that φ_i^+ and φ_i^- are homeomorphisms. **Hint**: Show that $\varphi_i^+ : U_i^+ \to D^n$ has a continuous inverse that is given by

$$D^n \to U_i^+,\ (y^1, \ldots, y^n) \mapsto \left(y^1, \ldots, y^i, \sqrt{1 - \sum_j (y^j)^2}, y^{i+1}, \ldots, y^n \right).$$

Use a similar argument for $\varphi_i^- : U_i^- \to D^n$.

(4) Show that for $0 \leq i \neq j \leq n$, the maps

$$(\varphi_j^+)(\varphi_i^+)^{-1} \colon D^n \to D^n$$

and

$$\varphi_j^-(\varphi_i^-)^{-1} \colon D^n \to D^n$$

are smooth. Conclude that $\{(U_i^+, \varphi_i^+), (U_i^-, \varphi_i^-)\}_{i=0}^n$ is a smooth atlas for S^n.

Solution.

(1) For $i = 0, \ldots, n$, the subset U_i^+ is the preimage of $(0, \infty)$ along the continuous function $x^i \colon S^n \subset \mathbb{R}^{n+1} \to \mathbb{R}$ and therefore open. Similarly, U_i^- is the preimage of $(-\infty, 0)$ along x^i.

(2) Since $0 \notin S^n$, given $p = (x^0, \ldots, x^n) \in S^n$, there is some $i \in \{0, \ldots, n\}$ such that $x^i \neq 0$. Thus either $p \in U_i^+$ or $p \in U_i^-$.

(3) The function

$$\psi_i^+ \colon D^n \to \mathbb{R}^{n+1},\ (y^1, \ldots, y^n) \mapsto \left(y^1, \ldots, y^i, \sqrt{1 - \sum_j (y^j)^2}, y^{i+1}, \ldots, y^n \right)$$

is well-defined and continuous since each of its components is continuous. Moreover, the identity

$$\sum_{j=1}^n (y^j)^2 + \left(\sqrt{1 - \sum_{j=1}^n (y^j)^2} \right)^2 = \sum_{j=1}^n (y^j)^2 + 1 - \sum_{j=1}^n (y^j)^2 = 1$$

shows that ψ_i^+ takes values in S^n. Since $\sqrt{1 - \sum_{j=1}^n (y^j)^2} > 0$, one moreover finds that ψ_i^+ takes values in U_i^+. Given $p = (x^0, \ldots, x^n) \in U_i^+$, we compute

$$\psi_i^+ \varphi_i^+(p) = \left(x^0, \ldots, x^{i-1}, \sqrt{1 - \sum_{j \neq i} (x^j)^2}, x^{i+1}, \ldots, x^n \right).$$

Since $p \in S^n$, one must have $\sum_i (x^i)^2 = 1$ and therefore

$$x^i = \sqrt{1 - \sum_{j \neq i} (x^j)^2},$$

hence we conclude that $\psi_i^+ \circ \varphi_i^+ = \mathrm{id}_{U_i^+}$. Conversely, given $q = (y^1, \ldots, y^n) \in D^n$, we have

$$\varphi_i^+ \psi_i^+(q) = (y^1, \ldots, y^n),$$

hence $\varphi_i^+ \circ \psi_i^+ = \mathrm{id}_{D^n}$. Therefore φ_i^+ is a homeomorphism. As for φ_i^-, essentially the same argumentation shows that an inverse is given by

$$\psi_i^-: D^n \to \mathbb{R}^{n+1},\ (y^1,\dots,y^n) \mapsto \left(y^1,\dots,y^i,-\sqrt{1-\sum_j (y^j)^2}, y^{i+1},\dots,y^n\right).$$

(4) Assume without loss of generality $i<j$. For $(y^1,\dots,y^n)\in D^n$, one computes

$$\varphi_j^+(\varphi_i^+)^{-1}(y^1,\dots,y^n) = (y^1,\dots,y^i,\sqrt{1-\sum_j (y^j)^2}, y^{i+1},\dots,y^{j-1},y^{j+1},\dots,y^n)$$

This function is smooth since both $\sqrt{-}\colon \mathbb{R}_{>0}\to\mathbb{R}$ and $\mathbb{R}^n \to \mathbb{R}, (y^1,\dots,y^n)\mapsto \sum_j (y^j)^2$ are smooth functions. The same argument shows that $\varphi_j^-(\varphi_i^-)^{-1}$ is smooth.

□

Exercise 4. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ be given by $F(x,y)=(x,y,xy)$ for $(x,y)\in\mathbb{R}^2$. Let (u,v,w) be the standard coordinates in $\mathbb{R}^3$. For $p=(x,y)\in\mathbb{R}^2$, compute $F_*(\frac{\partial}{\partial x}|_p)$ as a linear combination of $\frac{\partial}{\partial u}|_{F(p)}$, $\frac{\partial}{\partial v}|_{F(p)}$ and $\frac{\partial}{\partial w}|_{F(p)}$.

Solution. From the lectures we know that $F_*\colon T_p(\mathbb{R}^2)\to T_{F(p)}(\mathbb{R}^3)$ is represented by the Jacobian $J(p) = [\frac{\partial F^i}{x^j}(p)]$, where x^i is a basis element for $\mathbb{R}^2$. For $p=(x,y)$, the Jacobian matrix is given by

$$J(p) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ y & x \end{pmatrix}.$$

One can read off the coefficients for $F_*(\frac{\partial}{\partial x}|_p)$ from the first column of this matrix. Hence

$$F_*(\frac{\partial}{\partial x}|_p) = \frac{\partial}{\partial u}|_{F(p)} + y\frac{\partial}{\partial w}|_{F(p)}.$$

□

Exercise 5. As in Exercise 3, let $S^n \subset \mathbb{R}^{n+1}$ be the n-sphere. Let $p=(r^0,\dots,r^n)\in S^n$ be a point that is contained in U_0^+, i.e. that satisfies $r^0>0$. We have a coordinate map

$$\varphi_0^+\colon U_0^+ \to D^n \subset \mathbb{R}^n, \quad (r^0,\dots,r^n)\mapsto (r^1,\dots,r^n).$$

Let us write $\varphi_0^+ = (x^1,\dots,x^n)$, where x_j is the jth component of φ_0^+, i.e. the composition of φ_0^+ with the projection $s^j\colon \mathbb{R}^n\to\mathbb{R}$ onto the jth coordinate axis, where $(s^1,\dots,s^n)$ are the standard coordinates in $\mathbb{R}^n$. So for every $j=1,\dots,n$, the derivation $\frac{\partial}{\partial x^j}|_p$ is a tangent vector in S^n at the point p. Let $i\colon S^n \hookrightarrow \mathbb{R}^{n+1}$ be the inclusion. Then i is smooth, hence the differential $i_*\colon T_p(S^n)\to T_p(\mathbb{R}^{n+1})$ maps $\frac{\partial}{\partial x^j}|_p$ into $T_p(\mathbb{R}^{n+1})$.

(1) Compute $i_*(\frac{\partial}{\partial x^j}|_p)$ as a linear combination of $\frac{\partial}{\partial r^0}|_p,\dots,\frac{\partial}{\partial r^n}|_p$. Conclude that the map $i_*\colon T_p(S^n)\to T_p(\mathbb{R}^{n+1})$ is injective for *every* point $p\in S^n$.

(2) For arbitrary $p=(r^0,\dots,r^n)\in S^n$, prove that a tangent vector $X_p = \sum_i a_i \frac{\partial}{\partial r^i}|_p \in T_p(\mathbb{R}^{n+1})$ is contained in $T_p(S^n)$ if and only if $\sum_i a_i r^i = 0$.

Solution.

(1) First consider $p\in U_0^+$; later we will consider a general point $p\in S^n$. Also note that we write $T_{i(p)}(\mathbb{R}^n)$ as $T_p(\mathbb{R}^n)$ because $i(p)=p\in\mathbb{R}^n$. Since $i_*(\frac{\partial}{\partial x^j}|_p)\in T_p(\mathbb{R}^n)$, we can write it in terms of the basis $\{\frac{\partial}{\partial r^i}|_{i(p)}\}$

$$i_*(\frac{\partial}{\partial x^j}\Big|_p) = \sum_l a_{jl}\frac{\partial}{\partial r^l}\Big|_p, \quad a_{jl}\in\mathbb{R}.$$

We wish to find the coefficients a_{jl}. By evaluating both sides on r^k, $i_*(\frac{\partial}{\partial x^j}|_p)r^k = a_{jk}$. Recall that we defined partial derivatives on a manifold (such as S^n) via a chart and partial derivatives in Euclidean space. So by the definition of i_*,

$$\begin{aligned} i_*(\frac{\partial}{\partial x^j}\Big|_p)r^k &= \frac{\partial}{\partial x^j}\Big|_p (r^k \circ i) && \text{(by definition of } i_*) \\ &= \frac{\partial}{\partial s^j}\Big|_{\varphi_0^+(p)} (r^k \circ i \circ (\varphi_0^+)^{-1}) && \text{(by definition of the partial derivative on } S^n). \end{aligned}$$

To define the inverse $(\varphi_0^+)^{-1}$, recall that S^n is defined by points $(r^0, \dots, r^n) \in \mathbb{R}^n$ such that $\sum_i (r^i)^2 = 1$. In U_0^+, $r^0 > 0$. So $(\varphi_0^+)^{-1} \colon \mathbb{R}^n \to S^n$ is given by

$$(s^1, \dots, s^n) \mapsto \left(\sqrt{1 - \sum_i (s^i)^2}, s^1, \dots, s^n \right)$$

(also seen in Exercise 3). Hence the map

$$\mathbb{R}^n \xrightarrow{(\varphi_0^+)^{-1}} S^n \xhookrightarrow{i} \mathbb{R}^{n+1} \xrightarrow{r^k} \mathbb{R}$$

sends $(s^1, \dots, s^n) \mapsto s^k$ for $k \neq 0$ and $(s^1, \dots, s^n) \mapsto \sqrt{1 - \sum_i (s^i)^2}$ if $k = 0$. Note that for $k = 0$, the partial derivative is

$$\frac{\partial}{\partial s^j}\Big|_{\varphi_0^+(p)} \left(\sqrt{1 - \sum_i (s^i)^2} \right) = \frac{-s_j}{\sqrt{1 - \sum_i (s^i)^2}}\Bigg|_{\varphi_0^+(p)} = \frac{-r_j}{\sqrt{1 - \sum_i (r^i)^2}} = -\frac{r^j}{r^0}.$$

So by calculating $\frac{\partial}{\partial s^j}|_{\varphi_0^+(p)} (r^k \circ i \circ (\varphi_0^+)^{-1})$ for all k,

$$a_{jk} = i_*(\frac{\partial}{\partial x^j}|_p)r^k = \begin{cases} -\frac{r^j}{r^0} & \text{if } k = 0, \\ 1 & \text{if } j = k \text{ and } k \neq 0, \\ 0 & \text{if } j \neq k \text{ and } k \neq 0. \end{cases}$$

Therefore on U_0^+,

$$i_*(\frac{\partial}{\partial x^j}\Big|_p) = -\frac{r^j}{r^0} \frac{\partial}{\partial r^0}\Big|_p + \frac{\partial}{\partial r^j}\Big|_p.$$

Note that the matrix (a_{jk}) of coefficients is the Jacobian $J(p) = [\frac{\partial i^k}{\partial x^j}|_p]$ that represents i_*, where $i^k = r^k \circ i$ is the kth component of the inclusion map i. We therefore obtain

$$J(p) = \begin{pmatrix} -\frac{r^1}{r^0} & \cdots & -\frac{r^n}{r^0} \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}.$$

Note that $J(p)$ has full rank for all $p \in U_0^+$, hence the linear map $i_* \colon T_p(S^n) \to T_p(\mathbb{R}^{n+1})$ is injective for all points $p \in U_0^+$. Suppose now that $p \in S^n$ is arbitrarily chosen, i.e. not necessarily contained in U_0^+. Then there is $k \in \{0, \dots, n\}$ such that either $p \in U_k^+$ or $p \in U_k^-$. In the first case, the linear isomorphism $\tau_{0,k} \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ that transposes the coordinates r^0 and r^k restricts to a diffeomorphism $S^n \simeq S^n$ such that $\tau_{0,k}(p) \in U_0^+$. Moreover, the inclusion $i \colon S^n \hookrightarrow \mathbb{R}^{n+1}$ satisfies the identity $i = \tau_{0,k} i \tau_{0,k}$. On account of the chain rule and the fact that the pushforward of a diffeomorphism is an isomorphism of vector spaces, we conclude that $i_* \colon T_p(S^n) \to T_p(\mathbb{R}^{n+1})$ is injective. Finally, if $p \in U_k^-$, we may first use the map $\tau_{0,k}$ to transport p into U_0^- and then apply the isomorphism $\mathrm{inv}_0 \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $(r^0, \dots, r^n) \mapsto (-r^0, \dots, r^n)$ to send p into U_0^+. By the same argumentation as above, this implies that $i_* \colon T_p(S^n) \to T_p(\mathbb{R}^{n+1})$ is injective.

(2) By using the same argument as in part (1), we may assume without loss of generality that $p \in U_0^+$. Now $X_p \in T_p(\mathbb{R}^{n+1})$ is contained in $T_p(S^n)$ if and only if the vector $(a_0, \dots, a_n)$ is contained in the image of $J(\psi_0^+)(r^1, \dots, r^n)$. Given $(b_1, \dots, b_n) \in \mathbb{R}^n$, we compute

$$J(\psi_0^+)(r^1, \dots, r^n)(b_1, \dots, b_n) = \left(-\frac{1}{\sqrt{1 - \sum_{i=1}^n (r^i)^2}} \sum_j b_j r^j, b_1, \dots, b_n \right).$$

Hence $(a_0, \dots, a_n)$ is in the image of $J(\psi_0^+)(r^1, \dots, r^n)$ if and only if

$$a_0 = -\frac{1}{\sqrt{1 - \sum_{i=1}^n (r^i)^2}} \sum_{j=1}^n a_j r^j,$$

which is in turn equivalent to

$$a_0 \sqrt{1 - \sum_{i=1}^n (r^i)^2} = \sum_{j=1}^n a_j r^j.$$

Since $r^0 = \sqrt{1 - \sum_{i=1}^n (r^i)^2}$, this equation precisely means $\sum_{j=0}^n a_j r^j = 0$.

□

Exercise 6. Denote the standard coordinates of $\mathbb{R}^2$ by x, y and let

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

and

$$Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

be vector fields on $\mathbb{R}^2$. Find a 1-form ω on $\mathbb{R}^2 \setminus \{0\}$ that satisfies $\omega(X) = 1$ and $\omega(Y) = 0$.

Solution. The 1-form $\omega = -\frac{y}{x^2+y^2} y \mathrm{d}x + \frac{x}{x^2+y^2} \mathrm{d}y$ satisfies these conditions. □

Exercise 7. Let $x^1, y^1, \dots, x^n, y^n$ denote the standard coordinates of $\mathbb{R}^{2n}$.

(1) Show that the vector field $X = \sum_i -y^i \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial y^i}$ on $\mathbb{R}^{2n}$ is tangent to $S^{2n-1} \subset \mathbb{R}^{2n}$. That is, show that for every $p \in S^{2n-1}$ the tangent vector X_p is contained in $T_p(S^{2n-1}) \subset T_p(\mathbb{R}^{2n})$. Conclude that X restricts to a vector field on S^{2n-1}. **Hint**: Use Exercise 5.
(2) Show that the vector field X is nowhere vanishing on S^{2n-1}.
(3) Does there exist a nowhere vanishing vector field on an even-dimensional sphere?

Solution.

(1) By exercise 5, we need to show that for every $p = (r^1, s^1, \dots, r^n, s^n) \in S^{2n-1}$, the identity

$$\sum_i -r^i s^i + s^i r^i = 0$$

holds, which is clear.
(2) The vector field vanishes at a point $p \in S^{2n-1}$ when $X_p = 0$. Writing $p = (r^1, s^1, \dots, r^n, s^n)$, this precisely means that $-s^i = 0$ and $r^i = 0$ for all i, which is not possible unless $p = 0$.
(3) No, the hairy ball theorem tells us that this is not possible.

□

Exercise 8. Let $f \colon M \to N$ be a smooth map between manifolds. Let $\omega, \tau \in \Omega^1(N)$ and $g \in \mathrm{C}^\infty(N)$. Prove the following identities:

(1) $f^*(\omega + \tau) = f^*(\omega) + f^*(\tau)$;

(2) $f^*(g\omega) = f^*(g)f^*(\omega)$;
(3) $f^*(\mathrm{d}\omega) = \mathrm{d}f^*(\omega)$.

Hint: Evaluate either side of each equality at an arbitrary tangent vector X_p for $p \in M$.

Solution. Let X_p be an arbitrary tangent vector for $p \in M$. Then

(1) $f^*(\omega + \tau)(X_p) = (\omega + \tau)(f_*(X_p)) = \omega(f_*(X_p)) + \tau(f_*(X_p)) = f^*(\omega)(X_p) + f^*(\tau)(X_p)$;
(2) $f^*(g\omega)(X_p) = g\omega(f_*(X_p)) = g(f(p))\omega(f_*(X_p)) = f^*(g)(p)f^*(\omega)(X_p) = (f^*(g)f^*(\omega))(X_p)$;
(3) see proposition 19.5 in the book.

□

***Exercise 9.** For any integer $n \geq 1$, recall that the real projective space $\mathbb{P}^n(\mathbb{R})$ is defined as the set of 1-dimensional subspaces of $\mathbb{R}^{n+1}$. In this exercise we equip this set with the structure of a smooth manifold.

(1) Let $S^n = \{(x^0, \dots, x^n) \in \mathbb{R}^{n+1} \mid \sum_i x_i^2 = 1\} \subset \mathbb{R}^{n+1}$ be the unit sphere, and define an equivalence relation $\sim$ on S^n by declaring $x \sim y$ if $y = -x$. Show that there is a canonical bijection

$$\mathbb{P}^n(\mathbb{R}) \simeq S^n_{/\sim}.$$

By equipping S^n with the subspace topology and $S^n_{/\sim}$ with the quotient topology, we can regard $\mathbb{P}^n(\mathbb{R})$ as a topological space. This space is Hausdorff and second countable. We may parametrise the points in $\mathbb{P}^n(\mathbb{R})$ by equivalences classes of non-zero vectors $(x^0, \dots, x^n) \in \mathbb{R}^{n+1}$, where two such vectors are equivalent if and only if they are linearly dependent. We denote by $[x^0, \dots, x^n] \in \mathbb{P}^n(\mathbb{R})$ the point associated to the equivalence class of $(x^0, \dots, x^n) \in \mathbb{P}^n(\mathbb{R})$. This parametrisation is called the *homogeneous coordinate system*. For $i = 0, \dots, n$, let us define

$$U_i = \{[x^0, \dots, x^n] \mid x^i \neq 0\} \subset \mathbb{P}^n(\mathbb{R}).$$

(2) Show that U_i is open in $\mathbb{P}^n(\mathbb{R})$.
(3) Show that $\bigcup_i U_i = \mathbb{P}^n(\mathbb{R})$.
(4) Show that the assignment $[x^0, \dots, x^n] \mapsto (\frac{x^0}{x^i}, \dots, \frac{x^n}{x^i})$ determines a well-defined homeomorphism $\varphi_i \colon U_i \simeq \mathbb{R}^n$.
(5) Show that for any $0 \leq i \neq j \leq n$, the transition map

$$\varphi_j \varphi_i^{-1} \colon \mathbb{R}^n \to \mathbb{R}^n$$

is smooth.

As a consequence, we obtain a smooth atlas $\{(U_i, \varphi_i)\}_{i=0}^n$ for $\mathbb{P}^n(\mathbb{R})$, which gives this space the structure of a smooth manifold.

Solution.

(1) Let $\pi \colon S^n \to \mathbb{P}^n(\mathbb{R})$ be the map that sends $p \in S^n \subset \mathbb{R}^{n+1}$ to the 1-dimensional subspace that is spanned by the vector p. Then π is surjective since every 1-dimensional subspace of $\mathbb{R}^{n+1}$ has a basis vector of length 1, i.e. a basis vector that is contained in S^n. Moreover, $\pi(p) = \pi(1)$ if and only if p and q are linearly dependent, i.e. there is a $\lambda \in \mathbb{R}$ such that $p = \lambda q$. On account of $p, q \in S^n$, this is only possible if $\lambda = \pm 1$.
(2) On account of $\pi \colon S^n \to \mathbb{P}^n(\mathbb{R})$ being a quotient map, a subset $U \subset \mathbb{P}^n(\mathbb{R})$ is open if and only if $\pi^{-1}(U) \subset S^n$ is open. Note that in terms of homogeneous coordinates, the map π is given by the assignment $p = (x^0, \dots, x^n) \mapsto [x^0, \dots, x^n]$. Therefore $\pi^{-1}(U_i) = U_i^+ \cup U_i^-$ in the notation of exercise 3. Hence U_i is open.
(3) Since π is surjective we have $\bigcup_i U_i = \mathbb{P}^n(\mathbb{R})$ if and only if $\bigcup_i \pi^{-1}(U_i) = S^n$, hence the claim follows from part (2) of exercise 3.

(4) If $[x^1, \dots, x^n] \in U_i$ is given, note that since ${}^i \neq 0$ and since $[x^0, \dots, x^n] = [\lambda x^0, \dots, \lambda x^n]$ for every nonzero $\lambda \in \mathbb{R}$, we find

$$[x^0, \dots, x^n] = \left[\frac{x^0}{x^i}, \dots, \frac{x^{i-1}}{x^i}, 1, \frac{x^{i+1}}{x^i}, \dots, \frac{x^n}{x^i}\right].$$

As two representatives $(x^0, \dots, x^n)$ and $(y^0, \dots, y^n)$ of the same point in U_i are equal if and only if there is a $k \in \{0, \dots, n\}$ such that $x^k = y^k$, we find $\frac{x^j}{x^i} = \frac{y^j}{y^i}$ for every $j = 0, \dots, n$. Therefore the vector $(\frac{x^0}{x^i}, \dots, \frac{x^n}{x^i}) \in \mathbb{R}^n$ does not depend on the chosen representative of $[x^0, \dots, x^n]$, which implies that the map $[x^0, \dots, x^n] \mapsto (\frac{x^0}{x^i}, \dots, \frac{x^n}{x^i})$ is well-defined. The same argument shows that it is bijective: an inverse is given by

$$(z^1, \dots, z^n) \mapsto [z^1, \dots, z^i, 1, z^{i+1}, \dots, z^n].$$

In order to show that it is a homeomorphism, the fact that $\mathbb{P}^n\mathbb{R}$ is compact and $\mathbb{R}^n$ is Hausdorff implies that we only need to show that this map is continuous. By definition of the quotient topology, this is tantamount to showing that the induced map

$$U_i^+ \cup U_i^- \subset S^n \to \mathbb{R}^n, \quad (x^0, \dots, x^n) \mapsto \left(\frac{x^0}{x^i}, \dots, \frac{x^n}{x^i}\right)$$

is continuous, which is clear.

(5) Assume without loss of generality $i < j$. Then the map $\varphi_j \varphi_i^{-1}$ is given by

$$(z^1, \dots, z^n) \mapsto \left(\frac{z^1}{z^j}, \dots, \frac{z^i}{z^j}, \frac{1}{z^j}, \dots, \frac{z^{i+1}}{z^j}, \dots, \frac{z^n}{z^j}\right),$$

which is smooth since each of its coordinate maps are.

□