EXERCISE SHEET 2

***Exercise 1.** Let M be a smooth manifold and let $p \in M$ be a point. The goal of this exercise is to compare different definitions for the tangent space T_pM of M at p.

Let $\mathrm{Path}_p(M)$ be the set of smooth functions $\gamma\colon \mathbb{R} \to M$ that satisfy $\gamma(0) = p$. Recall that we define the stalk $C_p^\infty(M)$ as the set of equivalence classes of smooth maps $f\colon U \to \mathbb{R}$ where $U \subset M$ is an open neighbourhood of p and where two maps $f\colon U \to \mathbb{R}$ and $g\colon V \to \mathbb{R}$ are equivalent if they coincide on some open neighbourhood $W \subset U \cap V$ of p.

(1) Define

$$D_\gamma(f) = \frac{\mathrm{d}(f \circ \gamma)}{\mathrm{d}t}(0),$$

where $\gamma \in \mathrm{Path}_p(M)$ and where $f\colon U \to \mathbb{R}$ is a smooth map that is defined on an open neighbourhood $U \subset M$ of p. Show that D_γ determines a well-defined derivation

$$D_\gamma\colon C_p^\infty(M) \to \mathbb{R}.$$

Conclude that one obtains a map

$$D\colon \mathrm{Path}_p(M) \to T_p(M), \quad \gamma \mapsto D_\gamma.$$

(2) Show that the map $D\colon \mathrm{Path}(M) \to T_p(M)$ is surjective.

(3) Conclude that the map D descends to a bijection

$$\mathrm{Path}_p(M)_{/\sim} \simeq T_p(M)$$

where $\gamma \sim \delta$ if and only if there is a chart (U, φ) around p such that $\frac{\mathrm{d}(\varphi\circ\gamma)}{\mathrm{d}t}(0) = \frac{\mathrm{d}(\varphi\circ\delta)}{\mathrm{d}t}(0)$.

As a consequence of part (3), we may regard a tangent vector at $p \in M$ as an equivalence class of smooth paths passing through p. The correspondence between such paths and tangent vectors admits a geometric interpretation: locally, i.e. when M is an open subset of $\mathbb{R}^n$, every $\gamma \in \mathrm{Path}_p(M)$ is equivalent to exactly one straight line that passes through p, the line given by

$$\mathbb{R} \ni t \mapsto p + t\frac{\mathrm{d}\gamma}{\mathrm{d}t}(0).$$

The equivalence from (3) carries this line to the derivation that is given by the directional derivative along the vector $\frac{\mathrm{d}\gamma}{\mathrm{d}t}(0) \in \mathbb{R}^n$.

Recall that we denote by $C^\infty(M)$ the vector space of smooth functions $M \to \mathbb{R}$, where addition and scalar multiplication are defined pointwise. Note that this is a *ring* (in fact, an $\mathbb{R}$-algebra) when defining multiplication pointwise as well. Let $I_p \subset C^\infty(M)$ be the subspace of functions vanishing at p, i.e. those functions $f\colon M \to \mathbb{R}$ that satisfy $f(p) = 0$. This is an *ideal* in $C^\infty(M)$. Define $I_p^2 \subset I_p$ as the subspace generated by the set $\{fg \mid f, g \in I_p\}$. Explicitly, I_p^2 is comprised of those smooth functions $f\colon M \to \mathbb{R}$ that can be written as a linear combination

$$f = \sum_{i=1}^n f_i g_i$$

with $f_i, g_i \in I_p$ for all $i = 1, \ldots, n$.

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(4) Construct an isomorphism of vector spaces $T_p(M) \simeq (I_p/I_p^2)^\vee$. **Hint**: You may use without proof the smooth version of *Urysohn's lemma*: For any $K \subset U \subset M$ where K is closed and U is open, there is a smooth function $m\colon M \to \mathbb{R}$ satisfying $m|_K = 1$ and $M|_{M\setminus U} = 0$.

The isomorphism in part (4) gives rise to an identification of the cotangent space $T_p^\vee(M)$ with I_p/I_p^2. The geometric intuition behind this is as follows: if $f\colon M \to \mathbb{R}$ is a smooth function, Taylor's theorem implies that locally around p this function is given by a Taylor series

$$f(q) = f(p) + \sum_i \frac{\partial f}{\partial x^i}(p)(q^i - p^i) + o(|q-p|^2).$$

By applying the affine coordinate transformation $q \mapsto q + p$, we may assume without loss of generality $p = 0$. Now $f \in I_p$ precisely if the first term vanishes, and $f \in I_p^2$ precisely if the first two terms vanish. Consequently, every class $[f] \in I_p/I_p^2$ has a unique representative that is given by the map

$$\mathbb{R}^n \ni q \mapsto \sum_i \frac{\partial f}{\partial x^i}(p) q^i \in \mathbb{R},$$

i.e. by the covector $(\frac{\partial f}{\partial x^1}(p), \dots, \frac{\partial f}{\partial x^n}(p)) \in T_p^\vee(M)$.

Exercise 2. Let $X_1, \dots, X_n$ be n vector fields on an n-dimensional manifold M. Suppose that at $p \in M$, the vectors $(X_1)_p, \dots, (X_n)_p$ are linearly independent. Show that there is a chart $(V, x^1, \dots, x^n)$ around p such that $(X_i)_p = (\frac{\partial}{\partial x^i})_p$ for all $i = 1, \dots, n$.

***Exercise 3.** Let $S^n = \{(x^0, \dots, x^n) \in \mathbb{R}^{n+1} \mid \sum_i (x^i)^2 = 1\} \subset \mathbb{R}^{n+1}$ be the unit sphere. We equip S^n with the subspace topology of $\mathbb{R}^{n+1}$. For $i = 0, \dots, n+1$, define subsets

$$U_i^+ = \{(x^0, \dots, x^n) \in S^n \mid x^i > 0\} \subset S^n$$

and

$$U_i^- = \{(x^0, \dots, x^n) \in S^n \mid x^i < 0\} \subset S^n.$$

(1) Show that $U_i^+ \subset S^n$ and $U_i^- \subset S^n$ is open for every $i = 0, \dots, n$. **Hint**: Realise both U_i^+ and U_i^- as a preimage of an open interval in $\mathbb{R}^n$ along the coordinate projection $x^i\colon S^n \subset \mathbb{R}^{n+1} \to \mathbb{R}$.

(2) Show that $\bigcup_{i=0}^n (U_i^+ \cup U_i^-) = S^n$.

For every $i = 0, \dots n$, let

$$\varphi_i\colon S^n \to \mathbb{R}^n, \quad (x^0, \dots, x^n) \mapsto (x^0, \dots, x^{i-1}, x^{i+1}, \dots, x^n)$$

be the projection away from the ith coordinate, i.e. the projection onto the coordinate hyperplane that is perpendicular to the ith coordinate axis. Observe that $\varphi_i(U_i^+) = \varphi_i(U_i^-) = D^n$, where

$$D^n = \{(y^1, \dots, y^n) \in \mathbb{R}^n \mid \sum_j (y^j)^2 < 1\}$$

is the open unit disk in $\mathbb{R}^n$. By restricting φ_i to U_i^+ and U_i^-, we therefore obtain maps $\varphi_i^+ : U_i^+ \to D^n$ and $\varphi_i^- : U_i^- \to D^n$.

(3) Show that φ_i^+ and φ_i^- are homeomorphisms. **Hint**: Show that $\varphi_i^+ : U_i^+ \to D^n$ has a continuous inverse that is given by

$$D^n \to U_i^+, \ (y^1, \dots, y^n) \mapsto \left(y^1, \dots, y^i, \sqrt{1 - \sum_j (y^j)^2}, y^{i+1}, \dots, y^n \right).$$

Use a similar argument for $\varphi_i^- : U_i^- \to D^n$.

(4) Show that for $0 \leq i \neq j \leq n$, the maps

$$(\varphi_j^+)^{-1}\varphi_i^+ \colon D^n \to D^n$$

and

$$(\varphi_j^-)^{-1}\varphi_i^- \colon D^n \to D^n$$

are smooth. Conclude that $\{(U_i^+, \varphi_i^+), (U_i^-, \varphi_i^-)\}_{i=0}^n$ is a smooth atlas for S^n.

Exercise 4. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ be given by $F(x,y) = (x, y, xy)$ for $(x,y) \in \mathbb{R}^2$. Let (u, v, w) be the standard coordinates in $\mathbb{R}^3$. For $p = (x,y) \in \mathbb{R}^2$, compute $F_*(\frac{\partial}{\partial x}|_p)$ as a linear combination of $\frac{\partial}{\partial u}|_{F(p)}$, $\frac{\partial}{\partial v}|_{F(p)}$ and $\frac{\partial}{\partial w}|_{F(p)}$.

Exercise 5. As in Exercise 3, let $S^n \subset \mathbb{R}^{n+1}$ be the n-sphere. Let $p = (r^0, \ldots, r^n) \in S^n$ be a point that is contained in U_0^+, i.e. that satisfies $r^0 > 0$. We have a coordinate map

$$\varphi_0^+ \colon U_0^+ \to D^n \subset \mathbb{R}^n, \quad (r^0, \ldots, r^n) \mapsto (r^1, \ldots, r^n).$$

Let us write $\varphi_0^+ = (x^1, \ldots, x^n)$, where x_j is the jth component of φ_0^+, i.e. the composition of φ_0^+ with the projection $s^j \colon \mathbb{R}^n \to \mathbb{R}$ onto the jth coordinate axis, where $(s^1, \ldots, s^n)$ are the standard coordinates in $\mathbb{R}^n$. So for every $j = 1, \ldots, n$, the derivation $\frac{\partial}{\partial x^j}|_p$ is a tangent vector in S^n at the point p. Let $i\colon S^n \hookrightarrow \mathbb{R}^{n+1}$ be the inclusion. Then i is smooth, hence the differential $i_*\colon T_p(S^n) \to T_p(\mathbb{R}^{n+1})$ maps $\frac{\partial}{\partial x^j}|_p$ into $T_p(\mathbb{R}^{n+1})$.

(1) Compute $i_*(\frac{\partial}{\partial x^j}|_p)$ as a linear combination of $\frac{\partial}{\partial r^0}|_p, \ldots, \frac{\partial}{\partial r^n}|_p$. Conclude that the map $i_*\colon T_p(S^n) \to T_p(\mathbb{R}^{n+1})$ is injective for *every* point $p \in S^n$.
(2) For arbitrary $p = (r^0, \ldots, r^n) \in S^n$, prove that a tangent vector $X_p = \sum_i a_i \frac{\partial}{\partial r^i}|_p \in T_p(\mathbb{R}^{n+1})$ is contained in $T_p(S^n)$ if and only if $\sum_i a_i r^i = 0$.

Exercise 6. Denote the standard coordinates of $\mathbb{R}^2$ by x, y and let

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

and

$$Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

be vector fields on $\mathbb{R}^2$. Find a 1-form ω on $\mathbb{R}^2 \setminus \{0\}$ that satisfies $\omega(X) = 1$ and $\omega(Y) = 0$.

Exercise 7. Let $x^1, y^1, \ldots, x^n, y^n$ denote the standard coordinates of $\mathbb{R}^{2n}$.

(1) Show that the vector field $X = \sum_i -y^i\frac{\partial}{\partial x^i} + x^i\frac{\partial}{\partial y^i}$ on $\mathbb{R}^{2n}$ is tangent to $S^{2n-1} \subset \mathbb{R}^{2n}$. That is, show that for every $p \in S^{2n-1}$ the tangent vector X_p is contained in $T_p(S^{2n-1}) \subset T_p(\mathbb{R}^{2n})$. Conclude that X restricts to a vector field on S^{2n-1}. **Hint**: Use Exercise 5.
(2) Show that the vector field X is nowhere vanishing on S^{2n-1}.
(3) Does there exist a nowhere vanishing vector field on an even-dimensional sphere?

Exercise 8. Let $f\colon M \to N$ be a smooth map between manifolds. Let $\omega, \tau \in \Omega^1(N)$ and $g \in \mathrm{C}^\infty(N)$. Prove the following identities:

(1) $f^*(\omega + \tau) = f^*(\omega) + f^*(\tau)$;
(2) $f^*(g\omega) = f^*(g)f^*(\omega)$;
(3) $f^*(\mathrm{d}\omega) = \mathrm{d}f^*(\omega)$.

Hint: Evaluate either side of each equality at an arbitrary tangent vector X_p for $p \in M$.

***Exercise 9.** For any integer $n \geq 1$, recall that the real projective space $\mathbb{P}^n(\mathbb{R})$ is defined as the set of 1-dimensional subspaces of $\mathbb{R}^{n+1}$. In this exercise we equip this set with the structure of a smooth manifold.

(1) Let $S^n = \{(x^0, \dots, x^n) \in \mathbb{R}^{n+1} \mid \sum_i x_i^2 = 1\} \subset \mathbb{R}^{n+1}$ be the unit sphere, and define an equivalence relation $\sim$ on S^n by declaring $x \sim y$ if $y = -x$. Show that there is a canonical bijection

$$\mathbb{P}^n(\mathbb{R}) \simeq S^n_{/\sim}.$$

By equipping S^n with the subspace topology and $S^n_{/\sim}$ with the quotient topology, we can regard $\mathbb{P}^n(\mathbb{R})$ as a topological space. This space is Hausdorff and second countable. We may parametrise the points in $\mathbb{P}^n(\mathbb{R})$ by equivalences classes of non-zero vectors $(x^0, \dots, x^n) \in \mathbb{R}^{n+1}$, where two such vectors are equivalent if and only if they are linearly dependent. We denote by $[x^0, \dots, x^n] \in \mathbb{P}^n(\mathbb{R})$ the point associated to the equivalence class of $(x^0, \dots, x^n) \in \mathbb{P}^n(\mathbb{R})$. This parametrisation is called the *homogeneous coordinate system*. For $i = 0, \dots, n$, let us define

$$U_i = \{[x^0, \dots, x^n] \mid x^i \neq 0\} \subset \mathbb{P}^n(\mathbb{R}).$$

(2) Show that U_i is open in $\mathbb{P}^n(\mathbb{R})$.

(3) Show that $\bigcup_i U_i = \mathbb{P}^n(\mathbb{R})$.

(4) Show that the assignment $[x^0, \dots, x^n] \mapsto (\frac{x^0}{x^i}, \dots, \frac{x^n}{x^i})$ determines a well-defined homeomorphism $\varphi_i \colon U_i \simeq \mathbb{R}^n$.

(5) Show that for any $0 \leq i \neq j \leq n$, the transition map

$$\varphi_j \varphi_i^{-1} \colon \mathbb{R}^n \to \mathbb{R}^n$$

is smooth.

As a consequence, we obtain a smooth atlas $\{(U_i, \varphi_i)\}_{i=0}^n$ for $\mathbb{P}^n(\mathbb{R})$, which gives this space the structure of a smooth manifold.