EXERCISE SHEET 1

Exercise 1. At each point $p = (p^1, p^2, p^3) \in \mathbb{R}^3$, define a map $\omega_p \colon T_p(\mathbb{R}^3) \to \mathbb{R}$ by

$$\omega_p((v^1, v^2, v^3), (w^1, w^2, w^3)) = p^3 \det \begin{pmatrix} v^1, w^1 \\ v^2, w^2 \end{pmatrix}$$

for any $v = (v^1, v^2, v^3)$ and $w = (w^1, w^2, w^3)$ in $T_p(\mathbb{R}^3)$.

(1) Show that ω_p is an alternating bilinear map.

Thus $\omega \colon p \mapsto \omega_p$ defines a differential 2-form on $\mathbb{R}^3$.

(2) Write ω in terms of the standard basis $\mathrm{d}x^i \wedge \mathrm{d}x^j$ at each point.

Solution.

(1) Using the identification $T_p(\mathbb{R}^3) = \mathbb{R}^3$, the map ω_p is given by the composition

$$\mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{\mathrm{pr}_{1,2} \times \mathrm{pr}_{1,2}} \mathbb{R}^2 \times \mathbb{R}^2 \xrightarrow{\det} \mathbb{R} \xrightarrow{p^3 \cdot (-)} \mathbb{R}$$

in which $\mathrm{pr}_{1,2} \colon \mathbb{R}^3 \to \mathbb{R}^2$ denotes the projection onto the first two coordinates and where $p^3 \cdot (-)$ is the map given by multiplying with p^3. Since both $\mathrm{pr}_{1,2}$ and $p^3 \cdot (-)$ are linear maps, the claim follows once we show that det is alternating bilinear. To that end, recall that the determinant $\det \colon \mathbb{R}^{n^2} \to \mathbb{R}$ is defined by the formula

$$\det(v_1, \ldots, v_n) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} v_{i,\sigma(i)}$$

for any $v_1, \ldots, v_n \in \mathbb{R}^n$ where we denote by $v_{i,j}$ the jth entry of the vector v_i and where S_n denotes the group of permutations of $\{1, \ldots, n\}$. Given $w \in \mathbb{R}^n$, we thus find

$$\begin{aligned}
\det(v_1, \ldots, v_k + w, \ldots, v_n) &= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)(v_{k,\sigma(k)} + w_{k,\sigma(k)}) \prod_{i \neq k} v_{i,\sigma(i)} \\
&= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} v_{i,\sigma(i)} + \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) w_{k,\sigma(k)} \prod_{i \neq k} v_{i,\sigma(i)} \\
&= \det(v_1, \ldots, v_k, \ldots, v_n) + \det(v_1, \ldots, w, \ldots, v_n).
\end{aligned}$$

Similarly, one checks that for any $\lambda \in \mathbb{R}$ the identity

$$\det(v_1, \ldots, \lambda v_i, \ldots, v_n) = \lambda \det(v_1, \ldots, v_n)$$

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holds. Hence det is multilinear. To show that it is alternating, let $\tau \in S_n$ be an arbitrary permutation. Then

$$\begin{aligned}\det(v_{\tau(1)}, \dots, v_{\tau(n)}) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} v_{\tau(i),\sigma(i)} \\ &= \operatorname{sgn}(\tau) \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma \circ \tau^{-1}) \prod_{i=1}^{n} v_{\tau(i),\sigma(i))} \\ &= \operatorname{sgn}(\tau) \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma \circ \tau) \prod_{j=1}^{n} v_{j,(\sigma\circ\tau)(j)} \\ &= \operatorname{sgn}(\tau) \det(v_1, \dots, v_n),\end{aligned}$$

where in the penultimate step we substitute $i = \tau^{-1}(j)$. Hence det is alternating.

(2) Let

$$\omega_p = c_{1,2}\mathrm{d}x^1 \wedge \mathrm{d}x^2 + c_{1,3}\mathrm{d}x^1 \wedge \mathrm{d}x^3 + c_{2,3}\mathrm{d}x^2 \wedge \mathrm{d}x^3$$

be the expansion of ω_p in the standard basis. In order to find the coefficients $c_{i,j} \in \mathbb{R}$, let $\{x_1, x_2, x_3\}$ be the standard basis in $T_p^{\mathbb{R}^3}$. We then find $c_{i,j} = \omega_p(x_i, x_j)$ because the identity $\mathrm{d}x^i \wedge \mathrm{d}x^j(x_i, x_j) = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta, implies that evaluating ω_p at (x_i, x_j) sets all terms but the one involving $c_{i,j}$ to zero. Explicitly, we therefore find

$$\begin{aligned}c_{1,2} &= p^3 \\ c_{1,3} &= 0 \\ c_{2,3} &= 0.\end{aligned}$$

Hence the desired expansion is $\omega_p = p^3\mathrm{d}x^1 \wedge \mathrm{d}x^2$.

□

Exercise 2. Consider the following three differential forms on $\mathbb{R}^3$:

(1) $$x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y$$

(2) $$xy^2z^3\mathrm{d}x + y\sin(xz)\mathrm{d}z$$

(3) $$\frac{\mathrm{d}x \wedge \mathrm{d}y + x\mathrm{d}y \wedge \mathrm{d}z}{x^2 + y^2 + z^2 + 1}.$$

(1) Find the exterior derivative for each of the forms.
(2) Evaluate the second form at the vector field X on $\mathbb{R}^3$ that is given by

$$X(x, y, z) = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}.$$

Solution.

(1) Taking the exterior derivative yields the following forms:
   (a) $3\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$
   (b) $-2xyz^3\mathrm{d}x \wedge \mathrm{d}y + (yz\cos(xz) - 3xy^2z^2)\mathrm{d}x \wedge \mathrm{d}z + \sin(xz)\mathrm{d}y \wedge \mathrm{d}z$
   (c) $-\frac{-x^2+y^2+(z-1)^2}{(x^2+y^2+z^2+1)^2}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$
(2) Evaluation yields the function $xy^3z^3 - yz\sin(xz)$ on $\mathbb{R}^3$.

□

Exercise 3. Let V be a vector space of dimension 3 with basis $\{e_1, e_2, e_3\}$, and dual basis $\{\alpha^1, \alpha^2, \alpha^3\}$. To a 1-covector $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$ on V,we associate the vector $v_\alpha = (a_1, a_2, a_3) \in \mathbb{R}^3$. To the 2-covector

$$\gamma = c_1\alpha^2 \wedge \alpha^3 + c_2\alpha^3 \wedge \alpha^1 + c_3\alpha^1 \wedge \alpha^2$$

on V, we associate the vector $v_\gamma = (c_1, c_2, c_3) \in \mathbb{R}^3$. Show that under this correspondence, the wedge product of 1-covectors corresponds to the cross product of vectors in $\mathbb{R}^3$: if α is as above and if $\beta = b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3$, then $v_{\alpha\wedge\beta} = v_\alpha \times v_\beta$.

Solution. The coefficient of $\alpha \wedge \beta$ at the basis element $\alpha^i \wedge \alpha^j$ (with $1 \leq i < j \leq 3$) is given by the number $\alpha \wedge \beta(e_i, e_j)$, see exercise 1 for an explanation. For the same reason there are identities $\alpha(e_i) = a_i$ and $\beta(e_i) = b_i$ for all $1 \leq i \leq 3$. By definition of the wedge product, one therefore computes

$$\alpha \wedge \beta(e_i, e_j) = \det \begin{pmatrix} \alpha(e_i) & \alpha(e_j) \\ \beta(e_i) & \beta(e_j) \end{pmatrix} = \det \begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix} = a_ib_j - a_jb_i.$$

One therefore obtains

$$v_{\alpha\wedge\beta} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).$$

On the other hand, the cross product of v_α and v_β is by definition

$$v_\alpha \times v_\beta = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

and is therefore equal to $v_{\alpha\wedge\beta}$. □

Exercise 4. Let α be a nonzero 1-covector and γ a k-covector on a finite-dimensional vector space V. Show that $\alpha \wedge \gamma = 0$ if and only if $\gamma = \alpha \wedge \beta$ for some $(k-1)$-covector β on V.

Solution. If $\gamma = \alpha \wedge \beta$ for some $(k-1)$-covector β, one has $\alpha \wedge \gamma = \alpha \wedge \alpha \wedge \beta = 0$ on account of the identity

$$\alpha \wedge \alpha \wedge \gamma = -\alpha \wedge \alpha \wedge \gamma$$

that arises from flipping the first two arguments.

For the converse, suppose that $\alpha \wedge \gamma = 0$. Since $\alpha \neq 0$, we may find a basis $\{\alpha^1, \ldots, \alpha^n\}$ of $V^\vee$ such that $\alpha = \alpha^1$. We may therefore write $\gamma = \sum_{i_1<\cdots<i_k} \lambda_{i_1<\cdots<i_k}\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}$. Now $\alpha \wedge \gamma = 0$ implies

$$0 = \sum_{i_1<\cdots<i_k} \lambda_{i_1<\cdots<i_k}\alpha^1 \wedge \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}.$$

Whenever $i_1 = 1$ we have $\alpha^1 \wedge \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} = 0$, and since the set $\{\alpha^1 \wedge \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} \mid i_1 > 1\}$ is linearly independent in $A_{k+1}(V)$, this implies $\lambda_{i_1<\cdots<i_k} = 0$ for any $i_1 < \cdots < i_k$ with $i_1 > 1$. Thus

$$\gamma = \sum_{1<i_2<\cdots<i_k} \lambda_{1<i_2<\cdots<i_k}\alpha^1 \wedge \alpha^{i_2} \wedge \cdots \wedge \alpha^{i_k} = \alpha \wedge \left(\sum_{1<i_2<\cdots<i_k} \lambda_{1<i_2<\cdots<i_k}\alpha^{i_2} \wedge \cdots \wedge \alpha^{i_k} \right).$$

□

Exercise 5. Let V be a finite-dimensional vector space and let A be the alternating operator as defined in the lectures. Prove that $A(f \otimes A(g)) = l!A(f \otimes g)$ for every $f \in A_k(V)$ and any $g \in A_l(V)$.

Solution. By definition,

$$A(f \otimes A(g)) = \sum_{\sigma\in S_{k+l}} \operatorname{sgn}(\sigma)\sigma \left(\sum_{\tau\in S_l} \operatorname{sgn}(\tau) f \otimes \tau(g) \right) = \sum_{\sigma\in S_{k+l}} \sum_{\tau\in S_l} \operatorname{sgn}(\sigma)\sigma(f \otimes \tau(g))$$

where S_k is the group of permutations of $\{1,\dots,k\}$ and S_{k+l} is the group of permutations of $\{1,\dots,k+l\}$. There is an injective map $S_k \hookrightarrow S_{k+l}$ that is defined by sending $\tau \in S_l$ to the permutation

$$\{1,\dots,k+l\} = \{0,\dots,k\} \sqcup \{k+1,\dots,k+l\} \xrightarrow{\mathrm{id}\sqcup\tau} \{0,\dots,k\} = \{k+1,\dots,k+l\} \simeq \{1,\dots,k+l\},$$

i.e. to the map defined by $\mathrm{id}\sqcup\tau(i) = i$ for all $1 \leq i \leq k$ and $\mathrm{id}\sqcup\tau(i) = \tau(i-k)$ for all $k+1 \leq i \leq k+l$.

In view of this identification, we find $f \otimes \tau(g) = (\mathrm{id}\sqcup\tau)(f \otimes g)$. As a consequence,

$$A(f \otimes A(g)) = \sum_{\sigma \in S_{k+l}} \sum_{\tau \in S_l} \mathrm{sgn}(\sigma(\mathrm{id}\sqcup\tau))\sigma(\mathrm{id}\sqcup\tau)(f \otimes g).$$

Since composition with an arbitrary but fixed $\sigma \in S_{k+l}$ defines a bijection $S_{k+l} \simeq S_{k+l}$, for any $\tau \in S_l$ there is a $\mu \in S_{k+l}$ such that $\sigma(\mathrm{id}\sqcup\tau) = \mu$. We therefore find

$$A(f \otimes A(g)) = l! \sum_{\mu \in S_{k+l}} \mathrm{sgn}(\mu)\mu(f \otimes g) = l! A(f \otimes g),$$

as desired. □

***Exercise 6.** The goal of this exercise is to look at wedge products from a slightly different perspective. Let V be a finite-dimensional vector space. Let $n \geq 1$ be an integer. We begin with the following observation:

(1) Show that a multilinear map $\varphi\colon V^n \to \mathbb{R}$ is alternating if and only if $\varphi(v_1,\dots,v_n) = 0$ for every sequence $(v_1,\dots,v_n) \in V^n$ such that there are $i \neq j$ with $v_i = v_j$.

We define the *free* vector space on a set S as

$$F(S) = \bigoplus_{s \in S} \mathbb{R}.$$

The vector space $F(S)$ has a basis $\{e_s,\ s \in S\}$ where e_s is the element in $F(S)$ that corresponds to the sequence $(\lambda_s)_{s\in S}$ with $\lambda_s = 1$ and $\lambda_{s'} = 0$ for $s' \neq s$. Let $R \subset F(V^n)$ be the subspace that is spanned by the elements

$$e_{(v_1,\dots,v_i+w,\dots,v_n)} - e_{(v_1,\dots,w,\dots,v_n)} - e_{(v_1,\dots,v_i,\dots,v_n)} \text{ for all } v_1,\dots,v_n,w \in V \text{ and all } 1 \leq i \leq n,$$
$$e_{(v_1,\dots,\lambda v_i,v_n)} - \lambda e_{(v_1,\dots,v_i,\dots,v_n)} \text{ for all } v_1,\dots,v_n \in V, \text{ all } \lambda \in \mathbb{R} \text{ and all } 1 \leq i \leq n,$$
$$e_{(v_1,\dots,v_n)} \quad \text{for all } (v_1,\dots,v_n) \text{ such that there are } i \neq j \text{ with } v_i = v_j.$$

Set $\Lambda^n V = F(V^n)/R$. For $(v_1,\dots,v_n) \in V$ we denote by $v_1 \wedge \cdots \wedge v_n \in \Lambda^n V$ the image of $e_{(v_1,\dots,v_n)}$ under the quotient map $F(V^n) \twoheadrightarrow \Lambda^n V$. By construction, the set $\{v_1 \wedge \cdots \wedge v_n,\ (v_1,\dots,v_n) \in V^n\}$ generates $\Lambda^n V$.

(1) Show that the map $(v_1,\dots,v_n) \mapsto v_1 \wedge \cdots \wedge v_n$ defines an alternating multilinear map $\delta_V\colon V^n \to \Lambda^n V$.

(2) Let W be a vector space and let $\varphi\colon V^n \to W$ be an alternating multilinear map. Show that there exists a *unique* linear map $f\colon \Lambda^n V \to W$ such that $\varphi = f \circ \delta_V$.

In particular, one obtains a bijection

$$\Psi\colon\ \hom_{\mathbb{R}}(\Lambda^n V, k) \simeq A_n(V), \qquad f \mapsto f \circ \delta_V.$$

(3) Show that the map Ψ is an isomorphism of vector spaces.

Recall that a bilinear map $\varphi\colon V \times W \to \mathbb{R}$ is *non-degenerate* if $\varphi(v,w) = 0$ for all $w \in W$ implies $v = 0$. Any non-degenerate bilinear map φ gives rise to an inclusion $V \hookrightarrow W^\vee$ via the map $v \mapsto \varphi(v,-)$ which is an isomorphism if and only if $\dim W = \dim V$.

(4) Show that the map

$$\Lambda^n(V^\vee) \times \Lambda^n V \to \mathbb{R}, \qquad (f_1 \wedge \cdots \wedge f_n, v_1 \wedge \cdots \wedge v_n) \mapsto \det(f_i(v_j))$$

is well-defined, bilinear and non-degenerate.

As a consequence, one obtains an isomorphism $\mathrm{can}\colon \Lambda^n(V^\vee) \simeq A_n(V)$ of k-vector spaces.

(5) Show that the alternating map $\mathrm{can} \circ \delta_{V^\vee}\colon (V^\vee)^n \to \Lambda^n(V^\vee) \simeq A_n(V)$ recovers the wedge product as defined in the lectures.

Solution.

(1) If φ is alternating and if $(v_1, \dots, v_n)$ is a sequence in V^n such that there are $i \neq j$ with $v_i = v_j$, the permutation (ij) that transposes i and j does not change the tuple $(v_1, \dots, v_n)$, hence one finds

$$\varphi(v_1, \dots, v_n) = -\varphi(v_1, \dots, v_n)$$

which is only possible when $\varphi(v_1, \dots, v_n) = 0$.

Conversely, since every permutation σ can be written as a product of tranpositions, it suffices to show that φ satisfies the relation

$$\varphi(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = -\varphi(v_1, \dots, v_j, \dots, v_i, \dots, v_n)$$

for every $i \neq j$ and every tuple $(v_1, \dots, v_n)$ in V. By using multilinearity as well as the assumption on φ, we now find

$$\begin{aligned} 0 &= \varphi(v_1, \dots, v_i + v_j, \dots, v_i + v_j, \dots, v_n) \\ &= \varphi(v_1, \dots, v_i, \dots, v_j, \dots, v_n) + \varphi(v_1, \dots, v_j \dots, v_i \dots, v_n), \end{aligned}$$

which implies that the desired relation is satisfied.

(2) Given arbitrary $v_1, \dots, v_n, w \in V$ and any $1 \leq i \leq n$, the fact that the element

$$e_{(v_1, \dots, v_i + w, \dots, v_n)} - e_{(v_1, \dots, w, \dots, v_n)} - e_{(v_1, \dots, v_i, \dots, v_n)}$$

is contained in R precisely means the equation

$$(v_1 \wedge \dots \wedge (v_i + w) \wedge \dots \wedge v_n) - (v_1 \wedge \dots \wedge w \wedge \dots \wedge v_n) - (v_1 \wedge \dots \wedge v_i \wedge \dots \wedge v_n) = 0$$

holds in $\Lambda^n V$. In other words, one has

$$v_1 \wedge \dots \wedge (v_i + w) \wedge \dots \wedge v_n = (v_1 \wedge \dots \wedge w \wedge \dots \wedge v_n) + (v_1 \wedge \dots \wedge v_i \wedge \dots \wedge v_n).$$

Similarly, for any $\lambda \in \mathbb{R}$, the equation

$$v_1 \wedge \dots \wedge (\lambda v_i) \wedge \dots \wedge v_n = \lambda v_1 \wedge \dots \wedge v_i \wedge \dots \wedge v_n$$

holds since the element

$$e_{(v_1, \dots, \lambda v_i, v_n)} - \lambda e_{(v_1, \dots, v_i, \dots, v_n)}$$

is contained in R. Hence the map δ_V is multilinear. To show that it is alternating, it suffices to show that $\delta_V(v_1, \dots, v_n) = 0$ whenever there are $i \neq j$ with $v_i = v_j$, which is in turn a consequence of the assumption that R contains the corresponding element $e_{(v_1, \dots, v_n)} \in F(V^n)$.

(3) Since the elements $e_{(v_1, \dots, v_n)} \in F(V^n)$ form a basis of V^n, the assignment $e_{(v_1, \dots, v_n)} \mapsto \varphi(v_1, \dots, v_n)$ determines a unique linear map $g\colon F(V^n) \to W$. Moreover, since φ is multilinear and alternating, the generators for the subspace R are sent to 0, hence g descends to a linear map $f\colon \Lambda^n V \to W$. Note that f is uniquely determined by g since the quotient map $F(V^n) \to \Lambda^n V$ is surjective, hence f is uniquely determined by the assignment $e_{(v_1, \dots, v_n)} \mapsto \varphi(v_1, \dots, v_n)$, which moreover shows that $f \circ \delta_V = \varphi$ holds.

(4) We need to show that Ψ is linear. The computation

$$\begin{aligned}((f+g)\circ\partial_V)(v_1,\dots,v_n) &= (f+g)(v_1\wedge\cdots\wedge v_n)\\ &= f(v_1\wedge\cdots\wedge v_n)+g(v_1\wedge\cdots\wedge v_n)\\ &= (f\circ\partial_V)(v_1,\dots,v_n)+(g\circ\partial_V)(v_1,\dots,v_n)\end{aligned}$$

shows that Ψ is additive. Similarly, the computation

$$\begin{aligned}((\lambda f)\circ\partial_V)(v_1,\dots,v_n) &= (\lambda f)(v_1\wedge\cdots\wedge v_n)\\ &= \lambda(f(v_1\wedge\cdots\wedge v_n))\\ &= \lambda((f\circ\partial_V)(v_1,\dots,v_n))\end{aligned}$$

shows that Ψ commutes with scalar multiplication.

(5) Giving a bilinear map $\Lambda^n(V^\vee)\times\Lambda^n V\to\mathbb{R}$ is equivalent to giving a linear map $\Lambda^n(V^\vee)\to(\Lambda^n V)^\vee$, which by (2) is equivalent to giving an alternating map $(V^\vee)^n\to(\Lambda^n V)^\vee\simeq A_n(V)$, where we use the identification $(\Lambda^n V)^\vee\simeq A_n(V)$ from (3). In turn, any such map corresponds to a multilinear and alternating map $(V^\vee)^n\times V^n\to\mathbb{R}$. Hence we need to show that the map

$$(V^\vee)^n\times V^n\to\mathbb{R},\quad (f_1,\dots,f_n,v_1,\dots,v_n)\mapsto\det(f_i(v_j)).$$

is alternating and multilinear. Note that for any quadratic matrix M one has $\det M=\det M^\top$, hence det being multilinear and alternating implies immediately that the above map is multilinear and alternating as well. We still need to show that the resulting bilinear map

$$\Lambda^n(V^\vee)\times\Lambda^n V\to\mathbb{R},\qquad (f_1\wedge\cdots\wedge f_n, v_1\wedge\cdots\wedge v_n)\mapsto\det(f_i(v_j))$$

is non-degenerate. Let us thus assume that for a given n-tuple $(f_1,\dots,f_n)\in(V^\vee)^n$ one has $\det(f_i(v_j))=0$ for all $(v_1,\dots,v_n)$. Observe that since elements of the form $f_1\wedge\cdots\wedge f_n$ and $v_1\wedge\cdots\wedge v_n$ generate $\Lambda^n(V^\vee)$ and $\Lambda^n V$, respectively, it suffices to consider elements of that form. Then the set $\{f_1,\dots,f_n\}$ is linearly dependent. In fact, if $f_1,\dots,f_n$ were lienarly independent, there would be a tuple $(v_1,\dots,v_n)$ in V such that $f_i(v_j)=\delta_{ij}$ (simply by defining v_i to be the unique object in V that corresponds to the dual vector of f_i in $V^{\vee\vee}$ in view of the isomorphism $V\simeq V^{\vee\vee}$). But then $\det(f_i(v_j))=1$, a contradiction. Hence the $f_1,\dots,f_n$ are linearly dependent, but this necessarily means $f_1\wedge\cdots\wedge f_n=0$.

(6) The map can is constructed by composing the equivalence $\Lambda^n V^\vee\simeq(\Lambda^n V)^\vee$ that arises from the bilinear map in the previous part with the isomorphism $\Psi\colon(\Lambda^n V)^\vee\simeq A_n(V)$. Explicitly, can acts by sending elements in $\Lambda^n V^\vee$ of the form $f_1\wedge\cdots\wedge f_n$ to the alternating multilinear map

$$\mathrm{can}(f_1\wedge\cdots\wedge f_n)\colon V^n\to\mathbb{R},\quad (v_1,\dots,v_n)\mapsto\det(f_i(v_j)).$$

Therefore, the map $\mathrm{can}\circ\delta_V$ sends a tuple $(f_1,\dots,f_n)\in(V^\vee)^n$ to the multilinear alternating map

$$V^n\to\mathbb{R},\quad (v_1,\dots,v_n)\mapsto\det(f_i(v_j)),$$

which was shown in the lectures to be precisely the action of the wedge product of $(f_1,\dots,f_n)$.

□

***Exercise 7.** Let V be a finite-dimensional vector space and let $r\geq 1$ be an integer. The so-called *Grassmannian* $\mathrm{Grass}_r(V)$ is defined to be the set of r-dimensional subspaces of V, i.e. the set

$$\mathrm{Grass}_r(V)=\{r\text{-dimensional subspaces of } V\}.$$

Grassmannians generalize projective spaces: for $n \geq 1$, the projective space $\mathbb{P}^n(\mathbb{R})$ is given by the set of 1-dimensional subspaces of $\mathbb{R}^{n+1}$ and therefore by $\mathrm{Grass}_1(\mathbb{R}^{n+1})$. The goal of this exercise is to show that for any $r \geq 1$, the Grassmannian $\mathrm{Grass}_r(V^\vee)$ can be identified with a subset of the projective space $\mathbb{P}(A_r(V)) = \mathrm{Grass}_1(A_r(V))$ via the so-called *Plücker embedding*. Together with the previous exercise, this shows that the Grassmannian $\mathrm{Grass}_r(V)$ embeds into $\mathbb{P}(\Lambda^r V)$.

We say that an r-covector $\varphi \in A_r(V)$ is *decomposable* if there are r 1-covectors $\alpha_1, \dots, \alpha_r$ such that $\varphi = \alpha_1 \wedge \cdots \wedge \alpha_r$. Given an r-covector φ, we define

$$U_\varphi = \{\beta \in V^\vee \mid \varphi \wedge \beta = 0\} \subset V^\vee.$$

(1) Show that whenever $\varphi \neq 0$ is decomposable, the set U_φ defines an r-dimensional subspace of $V^\vee$.
(2) Let $U \subset V^\vee$ be an r-dimensional subspace. Show that there exists a decomposable r-covector $\varphi \neq 0$ such that $U = U_\varphi$.
(3) Show that if φ and ψ are two non-zero decomposable r-covectors such that $U_\varphi = U_\psi$, there is a nonzero scalar $\lambda \in \mathbb{R}$ such that $\varphi = \lambda\psi$.
(4) Conclude that there is an injective map

$$\mathrm{Grass}_r(V^\vee) \hookrightarrow \mathbb{P}(A_r(V))$$

whose image is given by those 1-dimensional subspaces of $A_r(V)$ that are spanned by a non-zero decomposable r-covector.

Solution.

(1) Let $\varphi = f_1 \wedge \cdots \wedge f_r$. Since by assumption $\varphi \neq 0$, the set $\{f_1, \dots, f_r\}$ must be linearly independent. Now $\beta \in U_\varphi$ precisely if $\varphi \wedge \beta = 0$, which is equivalent to $\{f_1 \dots, f_r, \beta\}$ being linearly dependent, hence to $\beta \in \mathrm{Span}(f_1, \dots, f_r)$. Therefore $U_\varphi = \mathrm{Span}(f_1, \dots, f_r)$ is an r-dimensional subspace of $V^\vee$.
(2) Choosing a basis $\{f_1, \dots, f_r\}$ of U, the argument in the previous part of the exercise shows that for $\varphi = f_1 \wedge \cdots \wedge f_r$ one has $U = U_\varphi$.
(3) If $\varphi = f_1 \wedge \cdots, f_r$ and $\psi = g_1 \wedge \cdots \wedge g_r$, we know from the previous arguments that $\{f_1, \dots, f_r\}$ and $\{g_1, \dots, g_r\}$ span the same r-dimensional subspace of $V^\vee$. Hence for every $i = 1 \dots, r$ there is a unique expansion

$$f_i = \sum_{j=1}^{n} \lambda_{i,j} g_j,$$

and the desired scalar λ is given by the determinant of the quadratic matrix $(\lambda_{i,j})_{1 \leq i,j \leq r}$.
(4) If U is an r-dimensional subspace of $V^\vee$, choose a basis $f_1, \dots, f_r$ of U and let $F(U) \in \mathbb{P}(A_r(V))$ be the 1-dimensional subspace of $A_r(V)$ that is spanned by $f_1 \wedge \cdots \wedge f_r$. By (3), the subspace $F(U)$ does not depend on the choice of basis $f_1, \dots, f_r$, hence one obtains a well-defined map

$$F\colon\ \mathrm{Grass}_r(V^\vee) \to \mathbb{P}(A_r(V)), \quad U \mapsto F(U).$$

By construction and (1), the image of F is equal to the subset of $\mathbb{P}(A_r(V))$ that is given by the subspaces of the form $\mathrm{Span}(f_1 \wedge \cdots \wedge f_r)$. By (1) and (2), the map $\varphi \mapsto U_\varphi$ from the image of F to $\mathrm{Grass}_r(V^\vee)$ defines an inverse of F, hence F is injective.

□