EXERCISE SHEET 1

Exercise 1. At each point $p = (p^1, p^2, p^3) \in \mathbb{R}^3$, define a map $\omega_p \colon T_p(\mathbb{R}^3) \to \mathbb{R}$ by

$$\omega_p((v^1, v^2, v^3), (w^1, w^2, w^3)) = p^3 \det \begin{pmatrix} v^1, w^1 \\ v^2, w^2 \end{pmatrix}$$

for any $v = (v^1, v^2, v^3)$ and $w = (w^1, w^2, w^3)$ in $T_p(\mathbb{R}^3)$.

(1) Show that ω_p is an alternating bilinear map.

Thus $\omega \colon p \mapsto \omega_p$ defines a differential 2-form on $\mathbb{R}^3$.

(2) Write ω in terms of the standard basis $\mathrm{d}x^i \wedge \mathrm{d}x^j$ at each point.

Exercise 2. Consider the following three differential forms on $\mathbb{R}^3$:

$$x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y \tag{1}$$

$$xy^2z^3\mathrm{d}x + y\sin(xz)\mathrm{d}z \tag{2}$$

$$\frac{\mathrm{d}x \wedge \mathrm{d}y + x\mathrm{d}y \wedge \mathrm{d}z}{x^2 + y^2 + z^2 + 1}. \tag{3}$$

(1) Find the exterior derivative for each of the forms.

(2) Evaluate the second form at the vector field X on $\mathbb{R}^3$ that is given by

$$X(x, y, z) = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}.$$

Exercise 3. Let V be a vector space of dimension 3 with basis $\{e_1, e_2, e_3\}$, and dual basis $\{\alpha^1, \alpha^2, \alpha^3\}$. To a 1-covector $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$ on V,we associate the vector $v_\alpha = (a_1, a_2, a_3) \in \mathbb{R}^3$. To the 2-covector

$$\gamma = c_1\alpha^2 \wedge \alpha^3 + c_2\alpha^3 \wedge \alpha^1 + c_3\alpha^1 \wedge \alpha^3$$

on V, we associate the vector $v_\gamma = (c_1, c_2, c_3) \in \mathbb{R}^3$. Show that under this correspondence, the wedge product of 1-covectors corresponds to the cross product of vectors in $\mathbb{R}^3$: if α is as above and if $\beta = b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3$, then $v_{\alpha\wedge\beta} = v_\alpha \times v_\beta$.

Exercise 4. Let α be a nonzero 1-covector and γ a k-covector on a finite-dimensional vector space V. Show that $\alpha \wedge \gamma = 0$ if and only if $\gamma = \alpha \wedge \beta$ for some $(k-1)$-covector β on V.

Exercise 5. Let V be a finite-dimensional vector space and let A be the alternating operator as defined in the lectures. Prove that $A(f \otimes A(g)) = l!A(f \otimes g)$ for every $f \in A_k(V)$ and any $g \in A_l(V)$.

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***Exercise 6.** The goal of this exercise is to look at wedge products from a slightly different perspective. Let V be a finite-dimensional vector space. Let $n \geq 1$ be an integer. We begin with the following observation:

(1) Show that a multilinear map $\varphi\colon V^n \to \mathbb{R}$ is alternating if and only if $\varphi(v_1, \ldots, v_n) = 0$ for every sequence $(v_1, \ldots, v_n) \in V^n$ such that there are $i \neq j$ with $v_i = v_j$.

We define the *free* vector space on a set S as

$$F(S) = \bigoplus_{s \in S} \mathbb{R}.$$

The vector space $F(S)$ has a basis $\{e_s,\ s \in S\}$ where e_s is the element in $F(S)$ that corresponds to the sequence $(\lambda_s)_{s \in S}$ with $\lambda_s = 1$ and $\lambda_{s'} = 0$ for $s' \neq s$. Let $R \subset F(V^n)$ be the subspace that is spanned by the elements

$$e_{(v_1,\ldots,v_i+w,\ldots,v_n)} - e_{(v_1,\ldots,w,\ldots,v_n)} - e_{(v_1,\ldots,v_i,\ldots,v_n)} \text{ for all } v_1, \ldots, v_n, w \in V \text{ and all } 1 \leq i \leq n,$$
$$e_{(v_1,\ldots,\lambda v_i,v_n)} - \lambda e_{(v_1,\ldots,v_i,\ldots,v_n)} \text{ for all } v_1, \ldots, v_n \in V, \text{ all } \lambda \in \mathbb{R} \text{ and all } 1 \leq i \leq n,$$
$$e_{(v_1,\ldots,v_n)} \quad \text{for all } (v_1, \ldots, v_n) \text{ such that there are } i \neq j \text{ with } v_i = v_j.$$

Set $\Lambda^n V = F(V^n)/R$. For $(v_1, \ldots, v_n) \in V$ we denote by $v_1 \wedge \cdots \wedge v_n \in \Lambda^n V$ the image of $e_{(v_1,\ldots,v_n)}$ under the quotient map $F(V^n) \twoheadrightarrow \Lambda^n V$. By construction, the set $\{v_1 \wedge \cdots \wedge v_n,\ (v_1, \ldots, v_n) \in V^n\}$ generates $\Lambda^n V$.

(1) Show that the map $(v_1, \ldots, v_n) \mapsto v_1 \wedge \cdots \wedge v_n$ defines an alternating multilinear map $\delta_V\colon V^n \to \Lambda^n V$.
(2) Let W be a vector space and let $\varphi\colon V^n \to W$ be an alternating multilinear map. Show that there exists a *unique* linear map $f\colon \Lambda^n V \to W$ such that $\varphi = f \circ \delta_V$.

In particular, one obtains a bijection

$$\Psi\colon\ \hom_{\mathbb{R}}(\Lambda^n V, k) \simeq A_n(V), \qquad f \mapsto f \circ \delta_V.$$

(3) Show that the map Ψ is an isomorphism of vector spaces.

Recall that a bilinear map $\varphi\colon V \times W \to \mathbb{R}$ is *non-degenerate* if $\varphi(v, w) = 0$ for all $w \in W$ implies $v = 0$. Any non-degenerate bilinear map φ gives rise to an inclusion $V \hookrightarrow W^\vee$ via the map $v \mapsto \varphi(v, -)$ which is an isomorphism if and only if $\dim W = \dim V$.

(4) Show that the map

$$\Lambda^n(V^\vee) \times \Lambda^n V \to \mathbb{R}, \qquad (f_1 \wedge \cdots \wedge f_n, v_1 \wedge \cdots \wedge v_n) \mapsto \det(f_i(v_j))$$

is well-defined, bilinear and non-degenerate.

As a consequence, one obtains an isomorphism $\mathrm{can}\colon \Lambda^n(V^\vee) \simeq A_n(V)$ of k-vector spaces.

(5) Show that the alternating map $\mathrm{can} \circ \delta_{V^\vee}\colon (V^\vee)^n \to \Lambda^n(V^\vee) \simeq A_n(V)$ recovers the wedge product as defined in the lectures.

***Exercise 7.** Let V be a finite-dimensional vector space and let $r \geq 1$ be an integer. The so-called *Grassmannian* $\mathrm{Grass}_r(V)$ is defined to be the set of r-dimensional subspaces of V, i.e. the set

$$\mathrm{Grass}_r(V) = \{r\text{-dimensional subspaces of } V\}.$$

Grassmannians generalize projective spaces: for $n \geq 1$, the projective space $\mathbb{P}^n(\mathbb{R})$ is given by the set of 1-dimensional subspaces of $\mathbb{R}^{n+1}$ and therefore by $\mathrm{Grass}_1(\mathbb{R}^{n+1})$. The goal of this exercise is to show that for any $r \geq 1$, the Grassmannian $\mathrm{Grass}_r(V^\vee)$ can be identified with a subset of the projective space $\mathbb{P}(A_r(V)) = \mathrm{Grass}_1(A_r(V))$ via the so-called *Plücker embedding.* Together with the previous exercise, this shows that the Grassmannian $\mathrm{Grass}_r(V)$ embeds into $\mathbb{P}(\Lambda^r V)$.

We say that an r-covector $\varphi \in A_r(V)$ is *decomposable* if there are r 1-covectors $\alpha_1, \dots, \alpha_r$ such that $\varphi = \alpha_1 \wedge \cdots \wedge \alpha_r$. Given an r-covector φ, we define

$$U_\varphi = \{\beta \in V^\vee \mid \varphi \wedge \beta = 0\} \subset V^\vee.$$

(1) Show that whenever $\varphi \neq 0$ is decomposable, the set U_φ defines an r-dimensional subspace of $V^\vee$.

(2) Let $U \subset V^\vee$ be an r-dimensional subspace. Show that there exists a decomposable r-covector $\varphi \neq 0$ such that $U = U_\varphi$.

(3) Show that if φ and ψ are two non-zero decomposable r-covectors such that $U_\varphi = U_\psi$, there is a nonzero scalar $\lambda \in \mathbb{R}$ such that $\varphi = \lambda\psi$.

(4) Conclude that there is an injective map

$$\mathrm{Grass}_r(V^\vee) \hookrightarrow \mathbb{P}(A_r(V))$$

whose image is given by those 1-dimensional subspaces of $A_r(V)$ that are spanned by a non-zero decomposable r-covector.