Repetition lecture

Overall aim: Ext and Tor are two of the most important constructions in homological algebra. Some of the main goals of the course:

(1) Understand the construction of Ext and Tor

(2) Develop the necessary language and techniques to be able to work with Ext and Tor

Overview of the course:

$\left.\begin{array}{l}\text{(1) General categories}\\ \text{(2) Additive and abelian categories}\end{array}\right\}$ language for homological algebra

$\left.\begin{array}{l}\text{(3) Hom and } \otimes\\ \text{(4) Complexes and homology}\end{array}\right\}$ foundation for Ext and Tor

(5) Derived functors (Ext and Tor as examples)

(6) Triangulated categories (Nice interpretation of Ext)

Note: Section 35 from lecture notes not covered

1: General categories

Overall aim (in Sec. 1-2): Develop a language for homological algebra $\rightsquigarrow$ category theory (foundation for the rest of the course)

Definitions: Category, subcategory, (split) monomorphism, (split) epimorphism, isomorphism, functor, natural transformation, full, faithful, dense, equivalence

Examples: (1) Set, Grp, Ab, ModR

(2) G group $\rightsquigarrow \mathcal{C}_G$: $\operatorname{Ob}\mathcal{C}_G = \{*\}$

$\operatorname{Hom}_{\mathcal{C}_G}(*,*) = G$

(3) $(X, \leq)$ poset $\rightsquigarrow \mathcal{C}_{(X,\leq)}$ poset category

(4) $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$

Note: (1) Objects are not necessarily sets and morphisms are not necessarily functions

(2) In general, a morphism being both mono and

epi does <u>not</u> imply that it is an iso. However, this <u>is</u> true in <u>abelian</u> categories.

Consequence of the Yoneda lemma

<u>Thm. (Yoneda embedding):</u> Let $\mathcal{C}$ small category. Then there is a fully faithful functor

$$\begin{aligned} \mathcal{C} &\longrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set}) \\ X &\longmapsto \mathrm{Hom}_{\mathcal{C}}(-,X) \end{aligned}$$

<u>Theorem:</u> A functor is an equivalence if and only if it is full, faithful and dense.

<u>Definitions:</u> Adjoint pair, product, coproduct

2: Additive and abelian categories

<u>Definitions:</u> preadditive category, additive category, preabelian category, abelian category, zero object, biproduct, additive functor, kernel, cokernel, image, exact sequence, short exact sequence, pullback, pushout

Slogan: An abelian category is an additive category with kernels and cokernels in which the first isomorphism theorem holds.

Lemma: Let $f: X \to Y$ morphism in additive cat. $\mathcal{A}$.

(1) a) f mono $\Leftrightarrow$ $\operatorname{Ker} f = 0$

b) f epi $\Leftrightarrow$ $\operatorname{Coker} f = 0$

(2) a) If $\operatorname{Ker} f$ exists, then $\operatorname{Ker} f \to X$ is a mono.

b) If $\operatorname{Coker} f$ exists, then $Y \to \operatorname{Coker} f$ is an epi.

Note: "Anything" defined in terms of a universal property is unique (up to unique iso)

$\rightsquigarrow$ e.g. kernels, cokernels, products, coproducts, pullbacks, pushouts ...

Thm. (five lemma): Let $\mathcal{A}$ abelian category. Consider a commutative diagram

$$\begin{array}{ccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5
\end{array}$$

in $\mathcal{A}$ with exact rows.

(1) If f_1 is epi and f_2 and f_4 are monos, then f_3 is mono.

(2) If f_5 is mono and f_2 and f_4 are epi, then f_3 is epi.

In particular, if f_1, f_2, f_4 and f_5 are isos, then f_3 is iso.

Thm. (snake lemma): Let $\mathcal{A}$ abelian category. Consider a commutative diagram

$$\begin{tikzcd}
& \operatorname{Ker} f_1 \arrow[r] \arrow[d, tail] & \operatorname{Ker} f_2 \arrow[r] \arrow[d, tail] & \operatorname{Ker} f_3 \arrow[d, tail] \arrow[dddll, rounded corners] & \\
& A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & B_1 \arrow[r] \arrow[d, two heads] & B_2 \arrow[r] \arrow[d, two heads] & B_3 \arrow[d, two heads] & \\
& \operatorname{Coker} f_1 \arrow[r] & \operatorname{Coker} f_2 \arrow[r] & \operatorname{Coker} f_3 &
\end{tikzcd}$$

in $\mathcal{A}$ with exact rows. Then there exists a morphism $\operatorname{Ker} f_3 \xrightarrow{\partial} \operatorname{Coker} f_1$ such that

$$\operatorname{Ker} f_1 \to \operatorname{Ker} f_2 \to \operatorname{Ker} f_3 \xrightarrow{\partial} \operatorname{Coker} f_1 \to \operatorname{Coker} f_2 \to \operatorname{Coker} f_3$$

is exact.

Proof strategy: Diagram chasing!

3: Hom and $\otimes$

Definitions: (left/right) exact functor, projective/injective object, tensor product

Thm: Let $\mathcal{A}$ abelian and $X \in \mathcal{A}$. Then the functors

$$\operatorname{Hom}_{\mathcal{A}}(X, -) : \mathcal{A} \longrightarrow \mathrm{Ab}$$
$$\operatorname{Hom}_{\mathcal{A}}(-, X) : \mathcal{A}^{op} \longrightarrow \mathrm{Ab}$$

are left exact.

"Hom is left exact"

Thm: "Tensor products exist and are unique up to (unique) isomorphism."

Thm: "($\otimes$, Hom) is an adjoint pair."

Cor.: "$\otimes$ is right exact"

Computation: Be able to compute some basic tensor products

4: Complexes and homology

Definitions: Complex, morphism of complexes, $C(\mathcal{A})$, $\boxed{\text{homology}}$, enough projectives/injectives, projective/injective resolution, quasi-isomorphism, $K(\mathcal{A})$, $\mathrm{Cone}(f^\bullet)$

Thm. (long exact sequence of homology): Let $\mathcal{A}$ abelian and $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ a short exact sequence in $C(\mathcal{A})$. Then we have an exact sequence

$$\cdots \to H^{n-1}(C^\bullet) \to H^n(A^\bullet) \to H^n(B^\bullet) \to H^n(C^\bullet) \to H^{n+1}(A^\bullet) \to \cdots$$

Key ingredient in proof: Snake lemma

Motivation for $K(\mathcal{A})$: Taking projective resolutions does not give a functor $\mathcal{A} \longrightarrow C(\mathcal{A})$.

Note: Let f morphism in $C(\mathcal{A})$ abelian. Then

f iso in $C(\mathcal{A})$ $\Longrightarrow$ f homotopy equivalence (iso in $K(\mathcal{A})$) $\Longrightarrow$ f quasi-iso (iso in $D(\mathcal{A})$)

Thm.: Let $\mathcal{A}$ abelian and $f^\bullet$ morphism in $C(\mathcal{A})$. Then $f^\bullet$ is a quasi-iso if and only if $\mathrm{Cone}(f^\bullet)$ is exact.

Thm.: Let $\mathcal{A}$ abelian with enough projectives. Then taking projective resolutions gives a functor $p: \mathcal{A} \longrightarrow K(\mathcal{A})$.

5: Derived functors

Motivation: Let $F: \mathcal{A} \longrightarrow \mathcal{B}$ left exact functor. Two natural questions:

(1) Can we measure "how far" F is from being exact?

(2) Given a short exact sequence

$$0 \to A \to B \to C \to 0 \quad \text{in } \mathcal{A},$$

can we continue the sequence

$$0 \to FA \to FB \to FC$$

to the right in a natural way?

left exact

right exact

<u>Definitions</u>: $\mathbb{R}^n F$, $\mathbb{L}_n F$, Ext^n_R, Tor^R_n, projective/injective dimension, global dimension

enough injectives

$$\mathcal{A} \xrightarrow{F} \mathcal{B}$$

left exact

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\mathbb{R}^n F"] \arrow[d, "i"'] & \mathcal{B} \\ K(\mathcal{A}) \arrow[r, "F_K"] & K(\mathcal{B}) \arrow[u, "H^n"'] \end{tikzcd}$$

enough projectives

$$\mathcal{A} \xrightarrow{F} \mathcal{B}$$

right exact

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\mathbb{L}_n F"] \arrow[d, "p"'] & \mathcal{B} \\ K(\mathcal{A}) \arrow[r, "F_K"] & K(\mathcal{B}) \arrow[u, "H^{-n}"'] \end{tikzcd}$$

Thm. (long exact sequence of derived functors):

Let $\mathcal{A}$ abelian with enough injectives, $F: \mathcal{A} \to \mathcal{B}$ left exact and $0 \to A \to B \to C \to 0$ short exact sequence in $\mathcal{A}$. Then there is an exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & FA \arrow[r] & FB \arrow[r] & FC \arrow[dll, out=0, in=180] \\
 & \mathbb{R}^1FA \arrow[r] & \mathbb{R}^1FB \arrow[r] & \mathbb{R}^1FC \arrow[dll, out=0, in=180] \\
 & \mathbb{R}^2FA \arrow[r] & \cdots &
\end{tikzcd}
$$

Strategy in proof: Use horseshoe lemma and apply long exact sequence in homology.

Thm. (balancing Ext and Tor):

(1) $\mathrm{Ext}^n_{\mathcal{A}}(X,-)(Y) \simeq \mathrm{Ext}^n_{\mathcal{A}}(-,Y)(X)$

(2) $\mathrm{Tor}^R_n(M,-)(N) \simeq \mathrm{Tor}^R_n(-,N)(M)$

Computation: Be able to compute basic examples of Ext and Tor

6: Triangulated categories

Motivation: $K(\mathcal{A})$ is almost never abelian, but it is triangulated!

Def.: Triangulated category!

Lemma: Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ distinguished triangle in a triangulated category $\mathcal{T}$. Then:

(1) $g \circ f = 0$

(2) $h = 0 \Rightarrow f$ split mono and g split epi

Thm. (long exact Hom sequence):

Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ distinguished triangle in a triangulated category $\mathcal{T}$. Then there is an exact sequence $\forall X \in \mathcal{T}$

$$\cdots \to \mathrm{Hom}_{\mathcal{T}}(X,A) \to \mathrm{Hom}_{\mathcal{T}}(X,B) \to \mathrm{Hom}_{\mathcal{T}}(X,C) \to$$

$$\to \mathrm{Hom}_{\mathcal{T}}(X,A[1]) \to \mathrm{Hom}_{\mathcal{T}}(X,B[1]) \to \mathrm{Hom}_{\mathcal{T}}(X,C[1]) \to \cdots$$

Definition: $D(\mathcal{A})$, $\mathrm{YExt}^1_{\mathcal{A}}$

Motivation:

(1) Make quasi-isos become isos ($\Rightarrow pX \simeq X$)

(2) Make short exact sequences become distinguished triangles

(3) Get nice interpretations of $\mathrm{Ext}^n_{\mathcal{A}}$

Thm./examples: "$K(\mathcal{A})$ and $D(\mathcal{A})$ are triangulated categories"

Thm.: Let $\mathcal{A}$ abelian with enough projectives or enough injectives. Consider $X, Y \in \mathcal{A}$. Then

$$\mathrm{Ext}^n_{\mathcal{A}}(X,Y) \simeq \mathrm{Hom}_{D(\mathcal{A})}(X, Y[n])$$

$$\left(\mathrm{Ext}^1_{\mathcal{A}}(X,Y) \simeq \mathrm{Hom}_{D(\mathcal{A})}(X, Y[1]) \simeq \mathrm{YExt}^1_{\mathcal{A}}(X,Y) \right)$$