MA3204 - Exercise 5

1. Let P be a projective right R-module, I an injective right R-module and F a flat right R-module. Show that

   (a) $\mathrm{Ext}^n_R(P, -) = 0$ for all $n > 0$;

   (b) $\mathrm{Ext}^n_R(-, I) = 0$ for all $n > 0$;

   (c) $\mathrm{Tor}^R_n(F, -) = 0$ for all $n > 0$.

2. Let $m, n \in \mathbb{Z}$.

   (a) Compute $\mathrm{Tor}^{\mathbb{Z}}_i(\mathbb{Z}_n, \mathbb{Z}_m)$ for all i.

   (b) Compute $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_m)$ for all i.

3. Let B be an abelian group, let $A \subseteq B$ be a subgroup of B, and let $n > 0$ be a positive integer. We can say that $\mathrm{Tor}^{\mathbb{Z}}_1(B/A, \mathbb{Z}/n\mathbb{Z})$ measures the obstruction to $nA = \{n \cdot a \mid a \in A\}$ being equal to the intersection $A \cap nB$. In this exercise we investigate why this is the case. Prove the following:

   (a) The canonical inclusion $nA \to A \cap nB$ is an isomorphism if and only if the canonical map $A/nA \to B/nB$ is a monomorphism.

   (b) Applying the tensor product $- \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ to the exact sequence $0 \to A \to B \to B/A \to 0$, show that if $\mathrm{Tor}^{\mathbb{Z}}_1(B/A, \mathbb{Z}/n\mathbb{Z}) = 0$ then the map $nA \to A \cap nB$ is an isomorphism.

   Hint: In the second part, use the long exact sequence of left derived functors. It might moreover be useful to recall Exercise 4 b) from the previous exercise set.

4. Consider the ring $R = \mathbb{Z}/4\mathbb{Z}$ and consider $M = \mathbb{Z}/2\mathbb{Z}$ as an R-module (check that there is a well-defined module structure!).

(a) Find a projective resolution of M as an R-module.
Hint: show that there exists an exact sequence

$$0 \to M \to R \to M \to 0$$

(b) Compute $\operatorname{Ext}^n_R(M, M)$ for all $n \geq 0$.

(c) Compute $\operatorname{Tor}^R_n(M, M)$ for all $n \geq 0$.

5. Let $\mathcal{A}$ be an additive category, let $X \in \mathcal{A}$, and let

$$A^\bullet = (\cdots \xrightarrow{d^{-2}} A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots)$$

be a complex in $\mathcal{A}$. The complex $\operatorname{Hom}_{\mathcal{A}}(A^\bullet, X)$ is given by

$$\cdots \xrightarrow{-\circ d^1} \operatorname{Hom}_{\mathcal{A}}(A^1, X) \xrightarrow{-\circ d^0} \operatorname{Hom}_{\mathcal{A}}(A^0, X) \xrightarrow{-\circ d^{-1}} \operatorname{Hom}_{\mathcal{A}}(A^{-1}, X) \xrightarrow{-\circ d^{-2}} \cdots$$

Show that

$$Z^n \operatorname{Hom}_{\mathcal{A}}(A^\bullet, X) = \operatorname{Hom}_{\mathcal{C}(\mathcal{A})}(A^\bullet, X[n])$$
$$H^n \operatorname{Hom}_{\mathcal{A}}(A^\bullet, X) = \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(A^\bullet, X[n])$$

Hint: Show that a morphism $f^{-n}\colon A^{-n} \to X$ induces a morphism $A^\bullet \to X[n]$ if and only if f^{-n} is in the kernel of

$$\operatorname{Hom}_{\mathcal{A}}(A^{-n}, X) \xrightarrow{-\circ d^{-n-1}} \operatorname{Hom}_{\mathcal{A}}(A^{-n-1}, X)$$

and is null-homotopic if and only if it is in the image of

$$\operatorname{Hom}_{\mathcal{A}}(A^{-n+1}, X) \xrightarrow{-\circ d^{-n}} \operatorname{Hom}_{\mathcal{A}}(A^{-n}, X)$$

6. Let $f : A^\bullet \to B^\bullet$ be a morphism in $\mathbf{K}(\mathcal{A})$. Show that the following are equivalent.

(a) f factors through an exact complex.

(b) There exists a quasi-isomorphism $q : H^\bullet \to A^\bullet$ such that $f \circ q = 0$.

(c) $f \cdot \operatorname{id}_{A^\bullet}^{-1} = 0$ in $\mathbf{D}(\mathcal{A})$.

Hint: Use the triangulated structure of $\mathbf{K}(\mathcal{A})$.

7. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ be a distinguished triangle in a triangulated category. Show that if $h = 0$, then f is a split monomorphism and g is a split epimorphism.

   Hint: Apply (TR3) to a diagram with morphisms between the given triangle and the trivial triangle of X.