MA3204 - Exercise 4

1. (Closure properties of projectives) Let $\mathcal{A}$ be an abelian category. Show that the following hold:

   (a) The zero object in $\mathcal{A}$ is projective

   (b) If P and Q are projective in $\mathcal{A}$, then the biproduct $P \oplus Q$ is projective in $\mathcal{A}$.

   (c) If $\{P_i\}_{i \in I}$ is a collection of projective objects in $\mathcal{A}$, and if the coproduct $\coprod_{i \in I} P_i$ exists in $\mathcal{A}$, then $\coprod_{i \in I} P_i$ is projective in $\mathcal{A}$

   (d) If P is projective in $\mathcal{A}$ and $P \cong P_1 \oplus P_2$, then P_1 and P_2 are projective in $\mathcal{A}$.

2. Let $\mathbb{K}$ be a field, and let $\mathsf{Vect}_{\mathbb{K}}$ be the category of $\mathbb{K}$-vector spaces. Show that every object in $\mathsf{Vect}_{\mathbb{K}}$ is projective and injective.

3. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories. Consider an exact sequence

$$\cdots \to X^{-2} \xrightarrow{d^{-2}} X^{-1} \xrightarrow{d^{-1}} X^{0} \xrightarrow{d^{0}} X^{1} \xrightarrow{d^{1}} X^{2} \to \cdots$$

   in $\mathcal{A}$. Show that the sequence

$$\cdots \to FX^{-2} \xrightarrow{F(d^{-2})} FX^{-1} \xrightarrow{F(d^{-1})} FX^{0} \xrightarrow{F(d^{0})} FX^{1} \xrightarrow{F(d^{1})} FX^{2} \to \cdots$$

   is exact in $\mathcal{B}$.

   Hint: Show that a long exact sequence is glued together by short exact sequences. Then use that a functor is exact if and only if it preserves short exact sequences.

4. Show that

   (a) For any commutative ring R and any ideals I and J of R, we have $R/I \otimes_R R/J = R/(I+J)$.

   (b) For every right R-module M over a ring R, and every two-sided ideal I of R, we have $M \otimes_R R/I = M/MI$.

5. Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor between abelian categories. We say that F *reflects exactness* if whenever

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

   is exact, then the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

   is exact. Show that if F is fully faithful and exact, then it reflects exactness.

   Hint: First show that if F is exact, then it preserves images and kernels. Hence F applied to the canonical morphism $\operatorname{Im} f \to \operatorname{Ker} g$ *gives the canonical morphism* $\operatorname{Im} F(f) \to \operatorname{Ker} F(g)$. *Finally, use that F is fully faithful to show that it reflects isomorphisms, i.e. that $F(h)$ being an isomorphism implies h is an isomorphism.*