MA3204 - Exercise 3

1. Consider two exact sequences in an abelian category $\mathcal{A}$ as follows:

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \qquad 0 \to D \xrightarrow{h} E \xrightarrow{k} F \to 0$$

   Show that the following is an exact sequence

$$0 \to A \oplus D \xrightarrow{f \oplus h} B \oplus E \xrightarrow{g \oplus k} C \oplus F \to 0.$$

2. Let $F \colon \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Show that F is exact if and only if F preserves short exact sequences, i.e. if for any exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathcal{A}$, the sequence $0 \to FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \to 0$ is exact in $\mathcal{B}$.

3. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let (F, G) be an adjoint pair of additive functors $F \colon \mathcal{A} \to \mathcal{B}$ and $G \colon \mathcal{B} \to \mathcal{A}$. Show that F is right exact and G is left exact.

4. Let $f \colon R \to S$ be a ring morphism. Show that f induces a faithful functor $f^* \colon \operatorname{Mod} S \to \operatorname{Mod} R$.

5. Show that $\mathcal{A}$ is an abelian category if and only if $\mathcal{A}^{op}$ is an abelian category.

6. Let $\mathcal{A}$ be a category and let be $\mathcal{C}$ a small category. Show that the following hold:

   - If $\mathcal{A}$ is additive, then $\operatorname{Fun}(\mathcal{C}, \mathcal{A})$ is additive.
   - If $\mathcal{A}$ is abelian, then $\operatorname{Fun}(\mathcal{C}, \mathcal{A})$ is abelian.

- If $\mathcal{A}$ is abelian, then a sequence $F_1 \xrightarrow{\eta} F_2 \xrightarrow{\epsilon} F_3$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{A})$ is exact if and only it is pointwise exact, i.e.

  $$F_1(X) \xrightarrow{\eta_X} F_2(X) \xrightarrow{\epsilon_X} F_3(X)$$

  is exact in $\mathcal{A}$ for every $X \in \mathcal{C}$.

7. Finish the proof of the five lemma in $\mathrm{Mod}\, R$ (we showed the first part of the result in class) using diagram chasing methods (i.e. using elements).

8. Finish the proof of the snake lemma in $\mathrm{Mod}\, R$ using diagram chasing methods (i.e. using elements).