MA3204 - Exercise 2

1. Let **Ab** denote the category of abelian groups. Recall that a torsion abelian group is an abelian group where every element has finite order. We can hence define
$$t\colon \mathbf{Ab} \to \mathbf{Ab}$$
by sending an abelian group M to its largest torsion subgroup $t(M)$, and sending a morphism $M \to N$ to its restriction $t(M) \to t(N)$ to the torsion subgroups.

   (a) Check that t as defined above is well-defined and gives a functor $t\colon \mathbf{Ab} \to \mathbf{Ab}$.

   (b) Check that the inclusion $t(M) \to M$ gives a natural transformation $t \to \mathrm{Id}_{\mathbf{Ab}}$

   (c) Let $\mathcal{T}$ be the full subcategory of **Ab** consisting of all torsion abelian groups. Show that t gives a functor
   $$t\colon \mathbf{Ab} \to \mathcal{T}$$
   which is right adjoint to the inclusion functor $\mathcal{T} \to \mathbf{Ab}$

   Hint: For N an abelian group and M a torsion abelian group, show that any additive map $M \to N$ must factor through $t(N)$. Then show that the inclusion $t(N) \to N$ induces a natural isomorphism
   $$\mathrm{Hom}_{\boldsymbol{Ab}}(M, t(N)) \xrightarrow{\cong} \mathrm{Hom}_{\boldsymbol{Ab}}(M, N)$$

2. Let $\mathcal{C}$ be a small category, and let $\mathcal{D}$ be any category. Let $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ denote the collection of all functors $F\colon \mathcal{C} \to \mathcal{D}$, and for $F, G \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ let
$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G)$$

denote the collection of natural transformations $F \to G$. Show that $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is a category. This is called the *functor category* from $\mathcal{C}$ to $\mathcal{D}$. What are the isomorphisms in this category?

3. Give an example of a category that is preadditive, but not additive.

   Hint: Let R be a ring. Consider the category $\mathcal{C}_R$ with $\mathrm{Obj}\, \mathcal{C}_R = \{*\}$ (i.e. consists of one object) and $\mathrm{Hom}_{\mathcal{C}_R}(*, *) = R$. Let composition be given by multiplication in R.

4. Show that kernels and cokernels are unique up to unique isomorphism if they exist.

   Hint: Assume that there exists two different kernels of the same morphism and use the universal property.