$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "="] & B \arrow[r] \arrow[d, "\phi"] & C \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & A \arrow[r, "\binom{1}{0}"] & A \oplus C \arrow[r, "(0\ 1)"] & C \arrow[r] & 0
\end{tikzcd}$$

By the five lemma ϕ is an isomorphism. This proves the claim ▮

Lecture 19

Theorem: $\mathcal{A}$ abelian cat with enough proj, $A \in \mathcal{A}$, and $n \geq 0$ integer. The following are equivalent:

(1) $pd(A) \leq n$

(2) If $0 \to X \rightarrowtail P^{-(n-1)} \to P^{-(n-2)} \to \dots \to P^{-1} \to P^{0} \to A \to 0$ exact with P^{-i} proj $\forall\, 0 \leq i \leq n-1$, then X is projective

(3) $Ext^{n+1}_{\mathcal{A}}(A, A') = 0 \quad \forall A' \in \mathcal{A}$

(4) $Ext^{i}_{\mathcal{A}}(A, A') = 0 \quad \forall A' \in \mathcal{A} \quad \forall\, i \geq n+1$

Proof: Clearly (4) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1)

Assume (1) and let $P^{\cdot}$ be a proj resolution of length n. Then $Ext^{i}(A, A') \cong H^{i}(Hom_{\mathcal{A}}(P^{\cdot}, A'))$

But $Hom_{\mathcal{A}}(P^{\cdot}, A')$ must be concentrated in degrees $0, \dots, n$ so $H^i(\mathrm{Hom}(P_\bullet, A)) = 0 \ \forall i \geq n+1$

This shows $(1) \overset{(i)}{\Rightarrow} (4)$!

It remains to show $(3) \Rightarrow (2)$. So let

$0 \to X \to P^{-(n-1)} \to \dots \to P^{-1} \to P^0 \to A \to 0$ be exact.

Have exact sequence

$0 \to \Omega A \to P^0 \to A \to 0$ $\quad \Omega A$ - syzygy of A.

Define recursively $\Omega^i A$ to be the kernel of

$P^{-(i-1)} \to \Omega^{i-1} A$, so that we have exact sequences

$0 \to \Omega^i A \to P^{-(i-1)} \to \Omega^{i-1} A \to 0$

$\Omega^i A$ - a syzygy of $\Omega^{i-1} A$.

Then have $X = \Omega^n A$. By dimension-shifting

$0 = \mathrm{Ext}^{n+1}_{\mathcal{C}}(A, -) \cong \mathrm{Ext}^n_{\mathcal{C}}(\Omega A, -) \cong \mathrm{Ext}^{n-1}_{\mathcal{C}}(\Omega^2 A, -) \cong \dots \cong \mathrm{Ext}^1_{\mathcal{C}}(\Omega^n A, -)$

Hence suffices to show $\mathrm{Ext}^1_{\mathcal{C}}(X, -) = 0 \Rightarrow X$ projective.

Choose exact sequence

$0 \to \Omega X \xrightarrow{i} P \xrightarrow{p} X \to 0$

Applying $\mathrm{Hom}(-, \Omega X)$, we get an exact sequence

$0 \to \mathrm{Hom}_{\mathcal{C}}(X, \Omega X) \to \mathrm{Hom}_{\mathcal{C}}(P, \Omega X) \underset{\substack{\uparrow \\ \text{surjective}}}{\to} \mathrm{Hom}_{\mathcal{C}}(\Omega X, \Omega X) \to \underset{\substack{\| \\ 0}}{\mathrm{Ext}^1_{\mathcal{C}}(X, \Omega X)}$

Since $\mathrm{Hom}_{\mathcal{A}}(P, \Omega X) \to \mathrm{Hom}_{\mathcal{A}}(\Omega X, \Omega X)$ surjective, $\exists f: P \to \Omega X$ s.t. $f \circ i = \mathrm{id}_{\Omega X}$.

By previous lemma, $P \cong X \oplus \Omega X$, so X is a summand of a projective object and is therefore projective.

Dual result

<u>Theorem</u>: $\mathcal{A}$ abelian cat with enough inj, $A \in \mathcal{A}$ and $n \geq 0$ integer. The following are equivalent:

(1) $\mathrm{id}(A) \leq n$

(2) If $0 \to A \to I^0 \to \dots \to I^{n-1} \to Y \to 0$ exact with I^j injective for $0 \leq j \leq n-1$, then Y is injective.

(3) $\mathrm{Ext}^{n+1}_{\mathcal{A}}(A', A) = 0 \ \forall A' \in \mathcal{A}$

(4) $\mathrm{Ext}^i_{\mathcal{A}}(A', A) = 0 \ \forall A' \in \mathcal{A} \ \forall i \geq n+1$. ∎

<u>Definition</u>: $\mathcal{A}$ abelian cat with enough projectives or enough injectives. The <u>global dimension</u> of $\mathcal{A}$, denoted $\mathrm{gl.dim}\, \mathcal{A}$, is defined to be the smallest integer $n \geq 0$ s.t.

$\mathrm{Ext}^{n+1}(A,B)=0 \ \forall A,B\in\mathcal{A}$.

Write $\mathrm{gl.dim}\,\mathcal{A}=\infty$ if no such integer exists.

<u>Theorem</u> $\mathcal{A}$ abelian cat with enough projectives and enough injectives. Let $n\geq 0$ be an integer. The following are equivalent:

(1) $\mathrm{gl.dim}\,\mathcal{A}\leq n$

(2) $\mathrm{pd}\,A\leq n \ \forall A\in\mathcal{A}$

(3) $\mathrm{id}\,A\leq n \ \forall A\in\mathcal{A}$

(4) $\mathrm{Ext}^i_{\mathcal{A}}(A,B)=0 \ \forall i\geq n+1, \forall A,B\in\mathcal{A}$.

<u>Proof</u>: This follows from the previous theorem and its dual ■

<u>Examples</u>:

(1) K field, then $\mathrm{gl.dim}\,\mathrm{Mod}\,K=0$

(2) $\mathrm{gl.dim}\,\mathrm{Mod}\,\mathbb{Z}=1$

$\mathrm{gl.dim}\,\mathrm{Mod}\,K[x]=1$

More generally, R principal ideal domain (comm ring, no $\neq 0$ zero divisor, every ideal gen by a single elt).

Then gl.dim Mod $R = 1$

(3) gl.dim Mod $\mathbb{k}[x_1, \dots, x_n] = n$

<u>Proposition</u> $\mathcal{A}$ abelian cat with enough proj or enough inj, and let $A, C \in \mathcal{A}$. If $\mathrm{Ext}^1_{\mathcal{A}}(C, A) = 0$, then any short exact sequence $0 \to A \to B \to C \to 0$ is split.

<u>Proof</u>: Assume $\mathcal{A}$ has enough projectives (enough injectives is dual)

Choose an exact sequence

$0 \to \Omega C \to P \to C \to 0$ with P projective.

For any exact sequence $0 \to A \to B \to C \to 0$, can find comm diagram

$$
\begin{tikzcd}
0 \arrow[r] & \Omega C \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}
$$

since P is projective.

Since left hand square is a pushout,

$0 \to A \to B \to C \to 0$ is split

$\iff \Omega C \to A$ factors through $\Omega C \rightarrowtail P$ (check!)

By dimension shifting, have

$\mathrm{Ext}^1(C, A) \cong \mathrm{Coker}(\mathrm{Hom}(P, A) \to \mathrm{Hom}(\Omega C, A))$

Hence $\mathrm{Hom}(P, A) \to \mathrm{Hom}(\Omega C, A)$ is surjective,

so any morphism $\Omega C \to A$ factors through

$\Omega C \rightarrowtail P$