(ii) If (*) pushout and f mono, then (*) pullback by Corollary 2.2. Hence $\operatorname{Ker} f \cong \operatorname{Ker} u$.

f mono $\Rightarrow \operatorname{Ker} f \cong 0 \Leftrightarrow \operatorname{Ker} u \cong 0 \Rightarrow u$ mono ∎

Lecture 9

Some diagram lemmas

In ModR we can determine exactness using elements.

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

with $g \circ f = 0$ is exact if $\forall\, y \in Y$ such that $g(y) = 0$, there exists $x \in X$ s.t. $f(x) = y$.

Proofs are often easier when using elements. Things are more complicated for general abelian categories.

In this section we will only give proofs for ModR. However, all statements hold for general abelian categories.

Theorem (Five lemma): $\mathcal{A}$ abelian cat.

Let

$$\begin{tikzcd}
A_1 \arrow[r, "a_1"] \arrow[d, "f_1"'] & A_2 \arrow[r, "a_2"] \arrow[d, "f_2"'] & A_3 \arrow[r, "a_3"] \arrow[d, "f_3"'] & A_4 \arrow[r, "a_5"] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\
B_1 \arrow[r, "b_1"] & B_2 \arrow[r, "b_2"] & B_3 \arrow[r, "b_3"] & B_4 \arrow[r, "b_4"] & B_5
\end{tikzcd}$$

be a commutative diagram with exact rows. The following hold

(1) If f_1 is epi and f_2 and f_4 are mono, then f_3 is mono.

(2) If f_5 is mono and f_2 and f_4 are epi, then f_3 is epi.

In particular, if f_1, f_2, f_4 and f_5 are iso, then f_3 is iso.

Pf (For Mod R)

(1) Let $x_3 \in A_3$ with $f_3(x_3) = 0$

Consider $a_3(x_3)$. Since

$f_4 a_3(x_3) = b_3 f_3(x_3) = 0$ & f_4 mono

$\Rightarrow a_3(x_3) = 0$. By exactness $\exists x_2 \in A_2$

s.t. $a_2(x_2) = x_3$

Then $b_2 f_2(x_2) = f_3 a_2(x_2) = f_3(x_3) = 0$

Exactness at $B_2 \Rightarrow \exists\, y_1 \in B_1$ s.t. $b_1(y_1) = f_2(x_2)$

f_1 epimorphism $\Rightarrow \exists\, x_1 \in A_1$ s.t. $f_1(x_1) = y_1$

$$\begin{tikzcd}
x_1 \in A_1 \arrow[r, "a_1"] \arrow[d, "f_1"'] & x_2 \in A_2 \arrow[r, "a_2"] \arrow[d, "f_2"'] & x_3 \in A_3 \arrow[r, "a_3"] \arrow[d, "f_3"'] & A_4 \arrow[r, "a_4"] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\
f_1(x_1) = y_1 \in B_1 \arrow[r, "b_1"'] & B_2 \arrow[r, "b_2"'] & B_3 \arrow[r, "b_3"'] & B_4 \arrow[r, "b_4"'] & B_5
\end{tikzcd}$$

$x_3 \mapsto a_3(x_3) = 0$

Compare $a_1(x_1)$ and x_2:

$f_2(a_1(x_1)) = b_1 f_1(x_1) = b_1(y_1) = f_2(x_2)$

f_2 mono $\Rightarrow a_1(x_1) = x_2$.

But then $x_3 = a_2(x_2) = a_2 a_1(x_1) = 0$ ✓

Prove (2) yourself!

Note: $\mathcal{A}$ abelian. A comm square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "v"] \\ Z \arrow[r, "u"'] & W \end{tikzcd}$$

induces a comm diagram

$$\begin{tikzcd} \operatorname{Ker} f \arrow[r, tail, "i_f"] \arrow[d, dashed, "\exists!\,\alpha_g"'] & X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[r, "p_f"] \arrow[d, "v"] & \operatorname{Coker} f \arrow[d, dashed, "\exists!\,\beta_v"] \\ \operatorname{Ker} u \arrow[r, "i_u"] & Z \arrow[r, "u"'] & W \arrow[r, "p_u"] & \operatorname{Coker} u \end{tikzcd}$$

Since $u \circ g \circ i_f = v \circ f \circ i_f = 0$

$\Rightarrow g \circ i_f$ factors through $\operatorname{Ker} u$ uniquely via α_g.

β_v constructed similarly

__Theorem__ (Snake lemma) $\mathcal{A}$ abelian cat.
Consider comm diagram with exact rows and columns

$$\begin{tikzcd}
 & \operatorname{Ker} f_1 \arrow[r, dashed, "\alpha_{a_1}"'] \arrow[d, tail, "i_{f_1}"] & \operatorname{Ker} f_2 \arrow[r, dashed, "\alpha_{a_2}"'] \arrow[d, tail, "i_{f_2}"] & \operatorname{Ker} f_3 \arrow[d, tail, "i_{f_3}"] \arrow[dddll, dashed, rounded corners] & \\
 & A_1 \arrow[r, "a_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "a_2"] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & B_1 \arrow[r, "b_1"'] \arrow[d, two heads, "p_{f_1}"] & B_2 \arrow[r, "b_2"'] \arrow[d, two heads, "p_{f_2}"] & B_3 \arrow[d, two heads, "p_{f_3}"] & \\
 & \operatorname{Coker} f_1 \arrow[r, dashed, "\beta_{b_1}"'] & \operatorname{Coker} f_2 \arrow[r, dashed, "\beta_{b_2}"'] & \operatorname{Coker} f_3 &
\end{tikzcd}$$

Then $\exists$ morphism $\partial : \operatorname{Ker} f_3 \to \operatorname{Coker} f_1$ s.t. the sequence

$$\begin{tikzcd}
\operatorname{Ker} f_1 \arrow[r, "\alpha_{a_1}"] & \operatorname{Ker} f_2 \arrow[r, "\alpha_{a_2}"] & \operatorname{Ker} f_3 \arrow[dll, rounded corners, "\partial"] \\
\operatorname{Coker} f_1 \arrow[r, "\beta_{b_1}"'] & \operatorname{Coker} f_2 \arrow[r, "\beta_{b_2}"'] & \operatorname{Coker} f_3
\end{tikzcd}$$

is exact.

For proof, see typesett notes.

<u>Construction of ∂ for Mod R:</u>

$$
\begin{tikzcd}
 & & \ker f_3 \ni z \arrow[d, "i_{f_3}"] \\
A_1 \arrow[r, "a_1"] \arrow[d, "f_1"] & A_2 \ni x_2 \arrow[r, "a_2"] \arrow[d, "f_2"] & A_3 \arrow[d, "f_3"] \\
y_1 \in B_1 \arrow[r, "b_1"] \arrow[d, "p_{f_1}"'] & B_2 \arrow[r, "b_2"] & B_3 \\
\operatorname{Coker} f_1 & &
\end{tikzcd}
$$

$a_2(x_2) = i_{f_3}(z)$

$b_1(y_1) = f_2(x_2)$

Let $z \in \ker f_3$.

a_2 surjective $\Rightarrow \exists\, x_2 \in A$ s.t. $a_2(x_2) = i_{f_3}(z)$

$b_2 f_2(x_2) = f_3 a_2(x_2) = f_3 i_{f_3}(z) = 0$

Exactness at $B_2 \Rightarrow \exists\, y_1 \in B_1$ s.t.

$b_1(y_1) = f_2(x_2)$

Define $\partial(z) := p_{f_1}(y_1)$.

Need to check:

- ∂ well-defined map
- ∂ morphism of R-modules
- Exactness (diagram chasing when $\mathcal{A} = \mathrm{Mod}\,R$)

<u>Theorem</u> (Characterization of pullback and pushout)
$\mathcal{A}$ abelian cat. Consider comm diagram

$$\begin{tikzcd}
\operatorname{Ker} f \arrow[r, tail, "i_f"] \arrow[d, "\alpha_g"] & X \arrow[r, "f"] \arrow[d, "g"] \arrow[dr, phantom, "(*)"] & Y \arrow[r, "p_f"] \arrow[d, "v"] & \operatorname{Coker} f \arrow[d, "\beta_v"] \\
\operatorname{Ker} u \arrow[r, tail, "i_u"] & Z \arrow[r, "u"'] & W \arrow[r, "p_u"] & \operatorname{Coker} u
\end{tikzcd}$$

with exact rows. The following hold:

(1) $(*)$ is pullback if and only if α_g is an iso & β_v is a mono

(2) $(*)$ is a pushout if and only if α_g is an epi & β_v is an iso.

<u>Pf</u>: See notes ∎

Hom, projectives and injectives

$\mathcal{A}, \mathcal{B}$ abelian cats.

Properties of functors $F: \mathcal{A} \to \mathcal{B}$?

<u>Def</u>: An additive functor $F: \mathcal{A} \to \mathcal{B}$ is

(1) <u>left exact</u> if $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ exact in $\mathcal{A}$ $\Rightarrow$ $0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ exact in $\mathcal{B}$

(2) <u>right exact</u> if $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ exact in $\mathcal{A}$ $\Rightarrow$ $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \to 0$ exact in $\mathcal{B}$

(3) <u>exact</u> if it is left and right exact.

<u>Note</u>: The following hold: (check!)

- F left exact iff F preserves kernels
- F right exact iff F preserves cokernels
- The following are equivalent

- F is exact
- F is left exact and preserves epimorphism
- F is right exact and preserves monomorphisms

Remark: A contravariant functor F from $\mathcal{A}$ to $\mathcal{B}$ is said to be left or right exact if the associated functor

$$F: \mathcal{A}^{op} \to \mathcal{B}$$

is left or right exact, respectively.
Note that $F: \mathcal{A}^{op} \to \mathcal{B}$ left (right) exact iff $F: \mathcal{A} \to \mathcal{B}$ is right (left) exact.

Recall: $\mathcal{C}$ cat, $X \in \mathcal{C}$, have functors

$$Hom_{\mathcal{C}}(X, -): \mathcal{C} \to Set$$

$$Hom_{\mathcal{C}}(-, X): \mathcal{C}^{op} \to Set.$$

If $\mathcal{A}$ preadditive cat, $X \in \mathcal{A}$, then

$$Hom_{\mathcal{A}}(X, -): \mathcal{A} \to Ab$$

$$Hom_{\mathcal{A}}(-, X): \mathcal{A}^{op} \to Ab.$$