

(ii) If $(*)$ pushout and f mono,
then $(*)$ pullback, by Corollary 2.2.
Hence $\text{Ker } f \cong \text{Ker } u$.

f mono $\Rightarrow \text{Ker } f \hat{=} 0 \Leftrightarrow \text{Ker } u \hat{=} 0 \Rightarrow u$ mono

Lecture 9

Some diagram lemmas

In $\text{Mod } R$ we can determine exactness
using elements.

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

with $g \circ f = 0$ is exact if $\forall y \in Y$
such that $g(y) = 0$, there exists $x \in X$
s.t. $f(x) = y$.

Proofs are often easier when using
elements. Things are more complicated
for general abelian categories.

In this section we will only give
proofs for $\text{Mod } R$. However, all statements
hold for general abelian categories.

Theorem (Five lemma): \mathcal{A} abelian cat.

$$\begin{array}{ccccccccc} \text{Let} & A_1 & \xrightarrow{a_1} & A_2 & \xrightarrow{a_2} & A_3 & \xrightarrow{a_3} & A_4 & \xrightarrow{a_5} & A_5 \\ & f_1 \downarrow & & f_2 \downarrow & & f_3 \downarrow & & f_4 \downarrow & & f_5 \downarrow \\ & B_1 & \xrightarrow{b_1} & B_2 & \xrightarrow{b_2} & B_3 & \xrightarrow{b_3} & B_4 & \xrightarrow{b_4} & B_5 \end{array}$$

be a commutative diagram with exact rows. The following hold

(1) If f_1 is epi and f_2 and f_4 are mono, then f_3 is mono.

(2) If f_5 is mono and f_2 and f_4 are epi, then f_3 is epi.

In particular, if f_1, f_2, f_4 and f_5 are iso, then f_3 is iso.

Pf (For Mod R)

(1) Let $x_3 \in A_3$ with $f_3(x_3) = 0$

Consider $a_3(x_3)$. Since

$$f_4 a_3(x_3) = b_3 f_3(x_3) = 0 \text{ \& } f_4 \text{ mono}$$

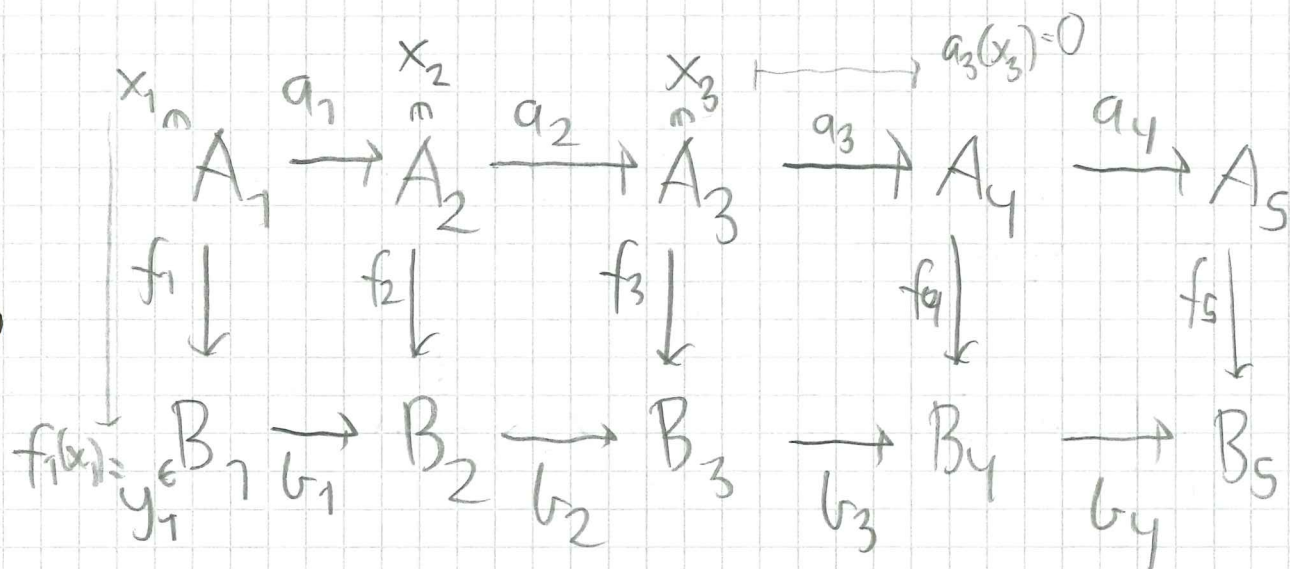
$\Rightarrow a_3(x_3) = 0$. By exactness $\exists x_2 \in A_2$

$$\text{s.t. } a_2(x_2) = x_3$$

Then $b_2 f_2(x_2) = f_3 a_2(x_2) = f_3(x_3) = 0$

Exactness at $B_2 \Rightarrow \exists y_1 \in B_1$ s.t. $b_1(y_1) = f_2(x_2)$

f_1 epimorphism $\Rightarrow \exists x_1 \in A_1$ s.t. $f_1(x_1) = y_1$



Compare $a_1(x_1)$ and x_2 :

$$f_2(a_1(x_1)) = b_1 f_1(x_1) = b_1(y_1) = f_2(x_2)$$

f_2 mono $\Rightarrow a_1(x_1) = x_2$.

But then $x_3 = a_2(x_2) = a_2 a_1(x_1) = 0 \quad \checkmark$

Prove (2) yourself!

Note: \mathcal{A} abelian. A comm square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \downarrow v \\ Z & \xrightarrow{u} & W \end{array}$$

induces a comm diagram

$$\begin{array}{ccccccc} \text{Ker } f & \xrightarrow{i_f} & X & \xrightarrow{f} & Y & \xrightarrow{p_f} & \text{Coker } f \\ \exists! \alpha_g \downarrow & & \downarrow g & & \downarrow v & & \downarrow \beta_v \\ \text{Ker } u & \xrightarrow{i_u} & Z & \xrightarrow{u} & W & \xrightarrow{p_u} & \text{Coker } u \end{array}$$

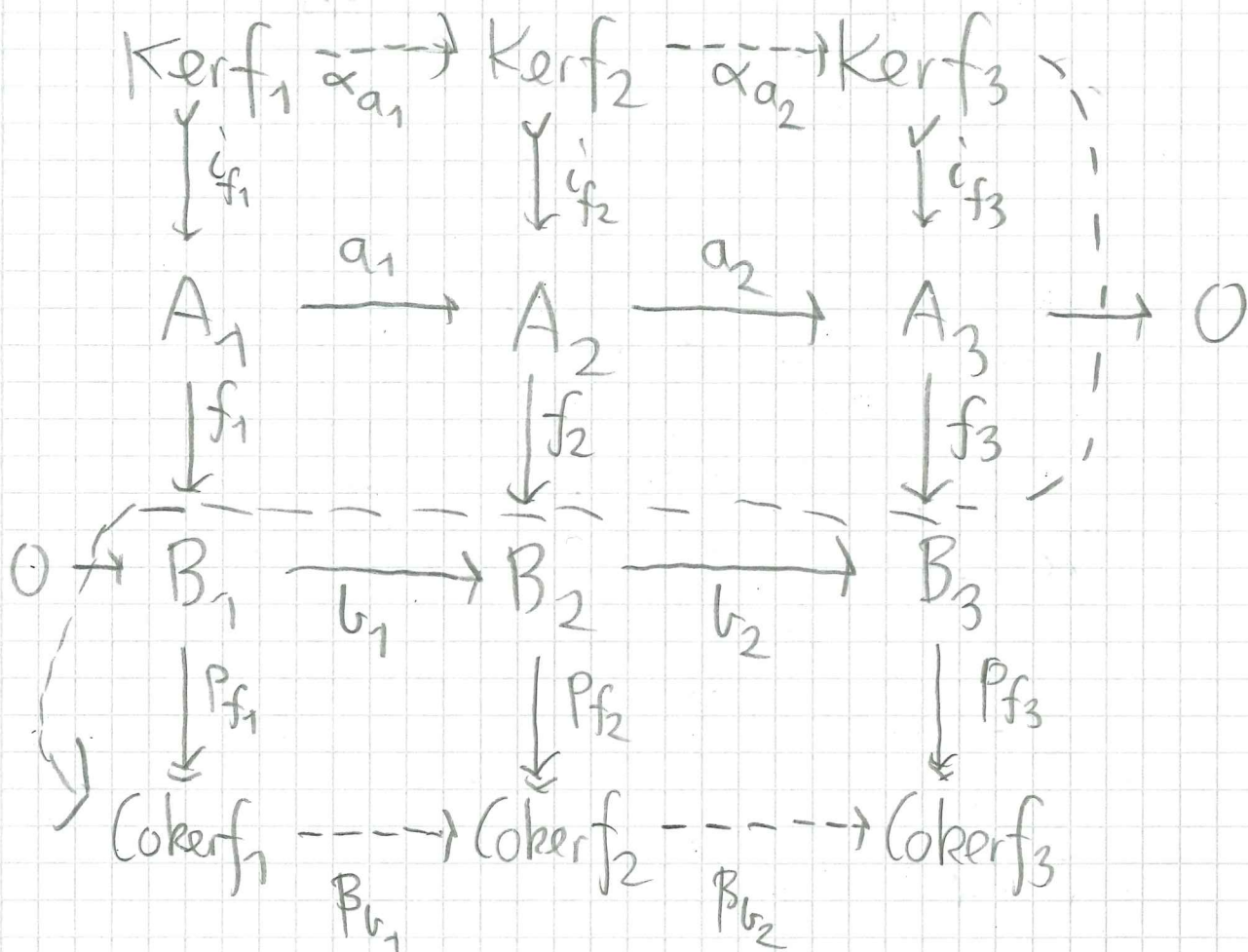
Since $u \circ g \circ i_f = v \circ f \circ i_f = 0$

$\Rightarrow g \circ i_f$ factors through $\text{Ker } u$

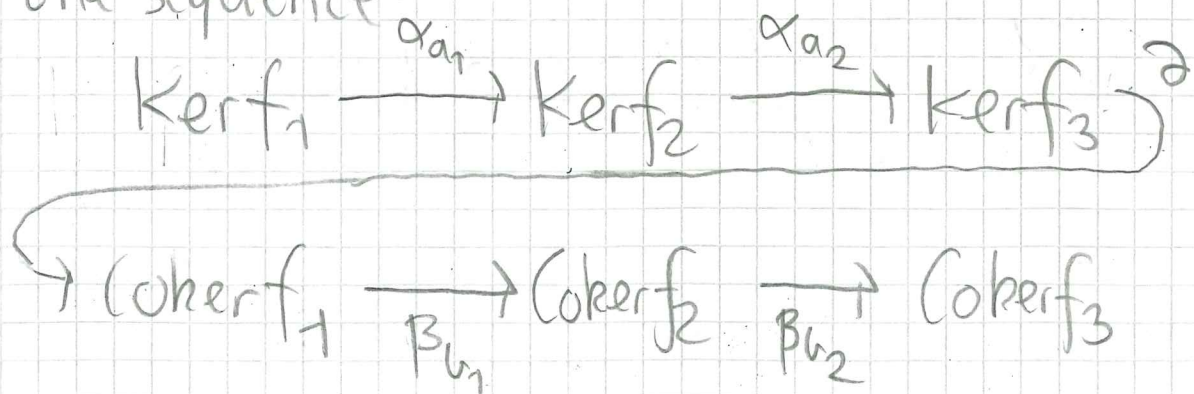
uniquely via α_g .

β_v constructed similarly

Theorem (Snake lemma) \mathcal{A} abelian cat.
 Consider comm diagram with exact rows and columns



Then \exists morphism $\partial: \text{Ker}f_3 \rightarrow \text{Coker}f_1$ s.t. the sequence

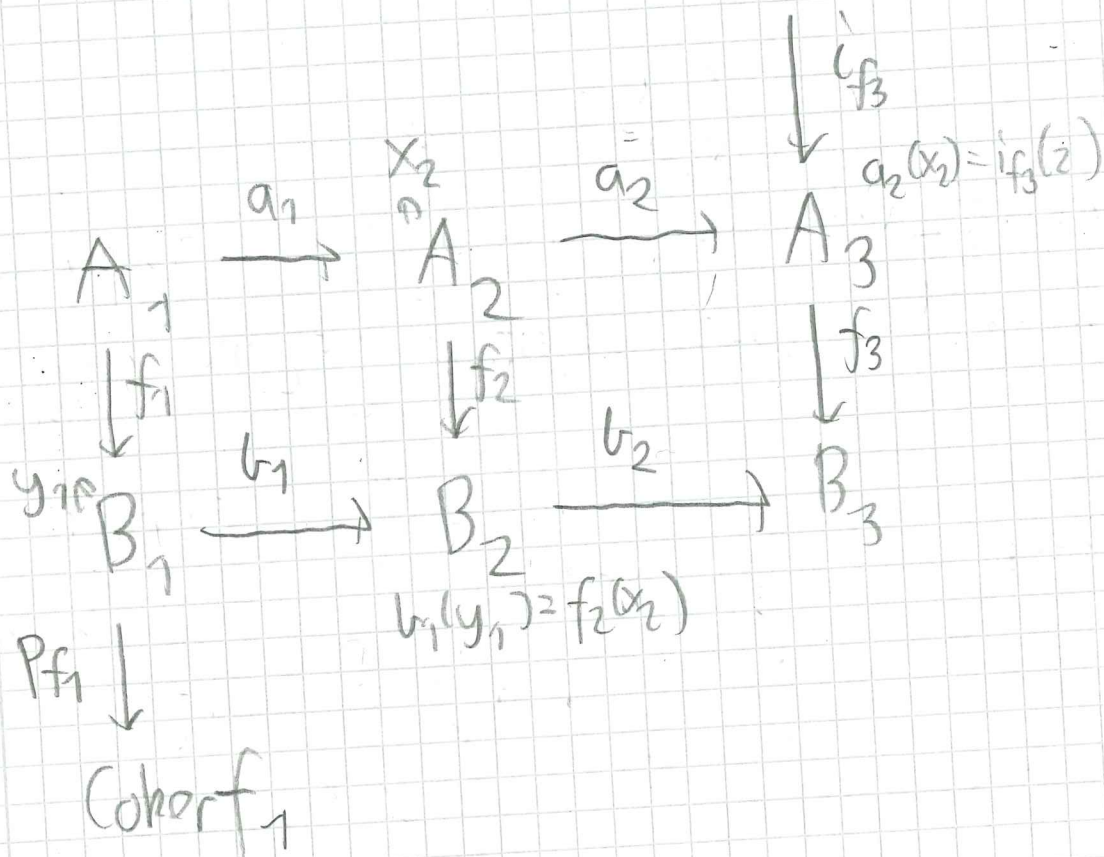


is exact.

For proof, see typeset notes.

Construction of ∂ for Mod R :

$\text{Ker } f_3 \cong Z$



Let $z \in \text{Ker } f_3$.
 a_2 surjective $\Rightarrow \exists x_2 \in A$ s.t. $a_2(x_2) = i_{f_3}(z)$

$$b_2 f_2(x_2) = f_3 a_2(x_2) = f_3 i_{f_3}(z) = 0$$

Exactness at $B_2 \Rightarrow \exists y_1 \in B_1$ s.t.

$$b_1(y_1) = f_2(x_2)$$

Define $\partial(z) := Pf_1(y_1)$.

Need to check:

- ∂ well-defined map
- ∂ morphism of R -modules
- Exactness (diagram chasing when $\mathcal{A} = \text{Mod } R$)

Theorem (Characterization of pullback and pushout)

\mathcal{A} abelian cat. Consider comm diagram

$$\begin{array}{ccccccc} \text{Ker } f & \xrightarrow{i_f} & X & \xrightarrow{f} & Y & \xrightarrow{P_f} & \text{Coker } f \\ \downarrow \alpha_g & & \downarrow g & (*) & \downarrow v & & \downarrow \beta_v \\ \text{Ker } u & \xrightarrow{i_u} & Z & \xrightarrow{u} & W & \xrightarrow{P_u} & \text{Coker } u \end{array}$$

with exact rows. The following hold:

(1) $(*)$ is pullback if and only if α_g is an iso & β_v is a mono

(2) $(*)$ is a pushout if and only if α_g is an epi & β_v is an iso.

Pf: See notes.

Hom, projectives and injectives

\mathcal{A}, \mathcal{B} abelian cats.

Properties of functors $F: \mathcal{A} \rightarrow \mathcal{B}$?

Def: An additive functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is

(1) left exact if $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ exact in $\mathcal{A} \Rightarrow 0 \rightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ exact in \mathcal{B}

(2) right exact if $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ exact in $\mathcal{A} \Rightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \rightarrow 0$ exact in \mathcal{B}

(3) exact if it is left and right exact.

Note: The following hold: (check!)

- F left exact iff F preserves kernels
- F right exact iff F preserves cokernels
- The following are equivalent

- F is exact
- F is left exact and preserves epimorphism
- F is right exact and preserves monomorphisms

Remark: A contravariant functor F from \mathcal{A} to \mathcal{B} is said to be left or right exact if the associated functor $F: \mathcal{A}^{\text{op}} \rightarrow \mathcal{B}$

is left or right exact, respectively.

Note that $F: \mathcal{A}^{\text{op}} \rightarrow \mathcal{B}$ left (right) exact iff $F: \mathcal{A} \rightarrow \mathcal{B}$ is right (left) exact.

Recall: \mathcal{C} cat, $X \in \mathcal{C}$, have functors

$$\text{Hom}_{\mathcal{C}}(X, -): \mathcal{C} \rightarrow \text{Set}$$

$$\text{Hom}_{\mathcal{C}}(-, X): \mathcal{C}^{\text{op}} \rightarrow \text{Set}$$

If \mathcal{A} preadditive cat, $X \in \mathcal{A}$, then

$$\text{Hom}_{\mathcal{A}}(X, -): \mathcal{A} \rightarrow \text{Ab}$$

$$\text{Hom}_{\mathcal{A}}(-, X): \mathcal{A}^{\text{op}} \rightarrow \text{Ab}$$