Repetition

Long exact sequences with Ext and Tor.

(i) Let R be a ring and $N \in \mathrm{Mod}\,R$. The functor

$$- \otimes_R N : \mathrm{Mod}\,R \longrightarrow \mathrm{Ab}$$

is right exact. We have the left derived functor

$$\mathrm{Tor}_n^R(-,N) = \mathbb{L}_n(- \otimes_R N) : \mathrm{Mod}\,R \longrightarrow \mathrm{Ab}.$$

Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence in $\mathrm{Mod}\,R$. Applying the theorem from Lecture 16 gives a long exact sequence

$$\cdots \longrightarrow \mathrm{Tor}_{n+1}^R(A,N) \xrightarrow{\mathrm{Tor}_{n+1}^R(f,N)} \mathrm{Tor}_{n+1}^R(B,N) \xrightarrow{\mathrm{Tor}_{n+1}^R(g,N)} \mathrm{Tor}_{n+1}^R(C,N)$$

$$\longrightarrow \mathrm{Tor}_n^R(A,N) \xrightarrow{\mathrm{Tor}_n^R(f,N)} \mathrm{Tor}_n^R(B,N) \xrightarrow{\mathrm{Tor}_n^R(g,N)} \mathrm{Tor}_n^R(C,N)$$

$$\vdots$$

$$\longrightarrow \operatorname{Tor}_1^R(A,N) \xrightarrow{\operatorname{Tor}_1^R(A,f)} \operatorname{Tor}_1^R(B,N) \xrightarrow{\operatorname{Tor}_1^R(g,N)} \operatorname{Tor}_1^R(C,N)$$

$$\longrightarrow A\otimes_R N \xrightarrow{f\otimes_R N} B\otimes_R N \xrightarrow{g\otimes_R N} C\otimes_R N \longrightarrow 0$$

of abelian groups.

(ii) Let $\mathcal{A}$ be an abelian category with enough projectives and $X \in \mathcal{A}$. The functor

$$\operatorname{Hom}_{\mathcal{A}}(X,-) : \mathcal{A} \longrightarrow Ab$$

is left exact. We have the right derived functor

$$\operatorname{Ext}^n_{\mathcal{A}}(X,-) = \mathbb{R}^n \operatorname{Hom}_{\mathcal{A}}(X,-) : \mathcal{A} \longrightarrow Ab.$$

Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence in $\mathcal{A}$. Applying the dual of the theorem from Lecture 16 gives a long exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{A}}(X,A) \xrightarrow{\operatorname{Hom}_{\mathcal{A}}(X,f)} \operatorname{Hom}_{\mathcal{A}}(X,B) \xrightarrow{\operatorname{Hom}_{\mathcal{A}}(X,f)} \operatorname{Hom}_{\mathcal{A}}(X,C)$$

$$\longrightarrow \operatorname{Ext}^1_{\mathcal{A}}(X,A) \xrightarrow{\operatorname{Ext}^1_{\mathcal{A}}(X,f)} \operatorname{Ext}^1_{\mathcal{A}}(X,B) \xrightarrow{\operatorname{Ext}^1_{\mathcal{A}}(X,g)} \operatorname{Ext}^1_{\mathcal{A}}(X,C)$$

$$\vdots$$

$$\longrightarrow \mathrm{Ext}^n_{\mathcal{A}}(X,A) \xrightarrow{\mathrm{Ext}^n_{\mathcal{A}}(X,f)} \mathrm{Ext}^n_{\mathcal{A}}(X,B) \xrightarrow{\mathrm{Ext}^n_{\mathcal{A}}(X,g)} \mathrm{Ext}^n_{\mathcal{A}}(X,C)$$

$$\longrightarrow \mathrm{Ext}^{n+1}_{\mathcal{A}}(X,A) \xrightarrow{\mathrm{Ext}^{n+1}_{\mathcal{A}}(X,f)} \mathrm{Ext}^{n+1}_{\mathcal{A}}(X,B) \xrightarrow{\mathrm{Ext}^{n+1}_{\mathcal{A}}(X,g)} \mathrm{Ext}^{n+1}_{\mathcal{A}}(X,C) \longrightarrow \cdots$$

Use this to solve problems 7, 8 and 9 from Exercise Set 5.

A composition in derived categories

$\mathcal{A}$ - abelian category. Let

$$f \cdot q^{-1} = \begin{tikzcd} & E^\bullet \arrow[dl, "q"', "\text{qiso}"] \arrow[dr, "f"] & \\ A^\bullet & & B^\bullet \end{tikzcd} \in \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A^\bullet, B^\bullet).$$

The q-iso $q : E^\bullet \longrightarrow A^\bullet$ can be seen as the morphism

$$q \cdot \mathrm{id}_E^{-1} = \begin{tikzcd} & E^\bullet \arrow[dl, "\mathrm{id}_{E^\bullet}"'] \arrow[dr, "q"] & \\ E^\bullet & & A^\bullet \end{tikzcd} \in \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(E^\bullet, A^\bullet),$$

which we have agreed to denote simply q.

Then we claim

$$(F\cdot q^{-1})\circ q = F \quad \text{in } \mathcal{D}(\mathcal{A}).$$

We compute $(F\cdot q^{-1})\circ q$:

$$\begin{tikzcd}
 & & E^\bullet \arrow[dl, "\mathrm{id}_{E^\bullet}"'] \arrow[dr, "\mathrm{id}_{E^\bullet}"] & & \\
 & E^\bullet \arrow[dl, "\mathrm{id}_{E^\bullet}"'] \arrow[dr, "q"] & & E^\bullet \arrow[dl, "q"'] \arrow[dr, "f"] & \\
E^\bullet & & A^\bullet & & B^\bullet
\end{tikzcd}$$

We apply Ore I and, since composition is well-defined, we may choose any way we prefer to complete this square

Then $(f\cdot q^{-1})\circ q =$
$$\begin{tikzcd}
 & E^\bullet \arrow[dl, "\mathrm{id}_{E^\bullet}"'] \arrow[dr, "f"] & \\
E^\bullet & & B^\bullet
\end{tikzcd}
= F.$$