- If $\mathcal{A}$ has enough injectives and F is left exact then the <u>n'th right derived functor</u> of F is

$$\mathbb{R}^n F = H^n \circ F \circ i : \mathcal{A} \longrightarrow \mathcal{B}$$

- $\mathbb{L}_n F(A)$ is computed as follows:

- Choose proj resolution of A

$$P^\bullet = \quad \cdots \to P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0$$

- apply F pointwise

$$\cdots \to F(P^{-2}) \xrightarrow{F(d^{-2})} F(P^{-1}) \xrightarrow{F(d^{-1})} F(P^0) = F_{ch}(P_\bullet)$$

- Take homology in degree $-n$:

$$H^{-n} F_{ch}(P^\bullet)$$

<u>Lecture 16</u>

- $\mathbb{L}_n(f)$ for a morphism $f: A \to B$ is computed as follows:
- Choose proj resolutions $P^\bullet$ and $Q^\bullet$ of A and B, and a lift $f^\bullet : P^\bullet \to Q^\bullet$ of f
- Apply F pointwise to get

$$g^\bullet = F_{ch}(f^\bullet) : F_{ch}(P^\bullet) \to F_{ch}(Q^\bullet)$$

- Take homology in degree $-n$: $\mathbb{L}_n(f) = H^{-n}(g^\bullet)$

Remark: If F is a contravarian functor from $\mathcal{A}$ to $\mathcal{B}$, then we define the nth left or right derived functor of F to be the nth left or right derived functor of the covariant functor $F: \mathcal{A}^{op} \longrightarrow \mathcal{B}$

Lemma: let $F: \mathcal{A} \to \mathcal{B}$ be a right exact functor, and assume $\mathcal{A}$ has enough projectives. Then $\mathbb{L}_0 F$ is naturally isomorphic to F.

Proof: $A \in \mathcal{A}$, $\cdots P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0$ proj resolution of A. Then $A \cong \operatorname{Coker} d^{-1}$. By definition

$$\mathbb{L}_0 F(A) = H^0(\cdots F(P^{-2}) \xrightarrow{F(d^{-2})} F(P^{-1}) \xrightarrow{F(d^{-1})} F(P^0))$$

$$= \operatorname{Coker} F(d^{-1})$$

Since F is right exact,

$$\operatorname{Coker} F(d^{-1}) \cong F(\operatorname{Coker} d^{-1}) \cong F(A).$$

Lemma Let $F: \mathcal{A} \to \mathcal{B}$ be an exact functor, and assume $\mathcal{A}$ has enough projectives. Then

$$\mathbb{L}_n F = 0 \text{ for } n \neq 0.$$

Pf: A proj resolution $P^\bullet = \cdots P^{-2} \to P^{-1} \to P^0$ of $A \in \mathcal{A}$ is exact in all $\neq 0$ degrees. Now since F is exact, $F(P^\bullet) = \cdots \to F(P^{-2}) \to F(P^{-1}) \to F(P^0)$ is also exact in all $\neq 0$ degrees. Hence
$\mathbb{L}_n F(A) \cong H^{-n}(F(P^\bullet)) = 0 \ \forall n \neq 0$. ∎

We also have the dual results:

Lemma: Let $F: \mathcal{A} \to \mathcal{B}$ left exact functor and assume $\mathcal{A}$ has enough injectives. Then
(1) $\mathbb{R}^0 F$ is naturally isomorphic to F
(2) If F is exact, then $\mathbb{R}^n F = 0 \ \forall n \neq 0$. ∎

We now answer our motivating question:

Theorem: $\mathcal{A}$ abelian with enough projectives, $F: \mathcal{A} \to \mathcal{B}$ right exact. For any short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{A}$ there is a long exact sequence

$$\to \mathbb{L}_{n+1} F(C) \longrightarrow \mathbb{L}_n F(A) \xrightarrow{\mathbb{L}_n F(f)} \mathbb{L}_n F(B) \xrightarrow{\mathbb{L}_n F(g)} \mathbb{L}_n F(C) \to \cdots \cdots$$

$$\cdots \xrightarrow[\mathbb{L}_1 F(f)]{} \mathbb{L}_1 F(B) \xrightarrow[\mathbb{L}_1 F(g)]{} \mathbb{L}_1 F(C) \longrightarrow F(A) \xrightarrow[F(f)]{} F(B) \xrightarrow[F(g)]{} F(C)$$

Proof By horseshoe lemma can find comm diagram with exact rows

$$\begin{tikzcd}
 & & & 0 \arrow[d] \\
\cdots P^{-2} \arrow[r] \arrow[d, "\binom{1}{0}"] & P^{-1} \arrow[r] \arrow[d, "\binom{1}{0}"] & P^{0} \arrow[r] \arrow[d, "\binom{1}{0}"] & A \arrow[d, "f"] \\
\cdots P^{-2}\oplus Q^{-2} \arrow[r] \arrow[d, "(0,1)"] & P^{-1}\oplus Q^{-1} \arrow[r] \arrow[d, "(0,1)"] & P^{0}\oplus Q^{0} \arrow[r] \arrow[d, "(0,1)"] & B \arrow[d, "g"] \\
\cdots Q^{-2} \arrow[r] & Q^{-1} \arrow[r] & Q^{0} \arrow[r] & C \arrow[d] \\
 & & & 0
\end{tikzcd}$$

Applying F_{ch} to the proj resolutions, get

$$\begin{tikzcd}
F_{ch}(P^{\cdot}) = (\cdots FP^{-2} \arrow[r] \arrow[d, "\binom{1}{0}"] & FP^{-1} \arrow[r] \arrow[d, "\binom{1}{0}"] & FP^{0} \arrow[d, "\binom{1}{0}"] \\
F_{ch}(R^{\cdot}) = (\cdots FP^{-2}\oplus FQ^{-2} \arrow[r] \arrow[d, "(0,1)"] & FP^{-1}\oplus FQ^{-1} \arrow[r] \arrow[d, "(0,1)"] & FP^{0}\oplus FQ^{0} \arrow[d, "(0,1)"] \\
F_{ch}(Q^{\cdot}) = (\cdots FQ^{-2} \arrow[r] & FQ^{-1} \arrow[r] & FQ^{0}
\end{tikzcd}$$

$\rightsquigarrow$ Get exact sequence $0 \to F_{Cn}(P^\bullet) \to F_{Cn}(R^\bullet) \to F_{Cn}(Q^\bullet) \to 0$ of complexes

The long exact sequence in homology gives the required result. ∎

Recall that we have left exact functors

$$\mathrm{Hom}_{\mathcal{A}}(X,-) : \mathcal{A} \longrightarrow Ab$$

$$\forall X \in \mathcal{A}$$

$$\mathrm{Hom}_{\mathcal{A}}(-,X) : \mathcal{A}^{op} \longrightarrow Ab$$

and right exact functor

$$- \otimes_R N : \mathrm{Mod}R \longrightarrow Ab \qquad R \text{ ring}$$

N left R-module

$$M \otimes_R - : \mathrm{Mod}R^{op} \longrightarrow Ab \qquad M \text{ right } R\text{-module}$$

Denote:

$$\mathrm{Ext}^n_{\mathcal{A}}(X,-) = \mathbb{R}^n \mathrm{Hom}_{\mathcal{A}}(X,-)$$

$$\mathrm{Ext}^n_{\mathcal{A}}(-,X) = \mathbb{R}^n \mathrm{Hom}_{\mathcal{A}}(-,X)$$

$$\mathrm{Tor}^R_n(-,N) = \mathbb{L}_n(- \otimes_R N)$$

$$\mathrm{Tor}^R_n(M,-) = \mathbb{L}_n(M \otimes_R -)$$

Will see later: $\mathrm{Ext}^n_{\mathcal{A}}(X,-)(Y) \cong \mathrm{Ext}^n_{\mathcal{A}}(-,Y)(X)$

$\mathrm{Tor}^R_n(M,-)(N) \cong \mathrm{Tor}^R_n(-,N)(M)$

$\rightsquigarrow$ Get exact sequence $0 \to F_{Cn}(P^\cdot) \to F_{Cn}(R^\cdot) \to F_{Cn}(Q^\cdot) \to 0$ of complexes

The long exact sequence in homology gives the required result. ▪

Recall that we have left exact functors

$$\mathrm{Hom}_{\mathcal{A}}(X,-): \mathcal{A} \longrightarrow \mathrm{Ab}$$

$$\mathrm{Hom}_{\mathcal{A}}(-,X): \mathcal{A}^{op} \longrightarrow \mathrm{Ab} \qquad \forall X \in \mathcal{A}$$

and right exact functor

$$- \otimes_R N : \mathrm{Mod}\,R \longrightarrow \mathrm{Ab} \qquad R \text{ ring}, \; N \text{ left } R\text{-module}$$

$$M \otimes_R - : \mathrm{Mod}\,R^{op} \longrightarrow \mathrm{Ab} \qquad M \text{ right } R\text{-module}$$

Denote:

$$\mathrm{Ext}^n_{\mathcal{A}}(X,-) = \mathbb{R}^n \mathrm{Hom}_{\mathcal{A}}(X,-)$$
$$\mathrm{Ext}^n_{\mathcal{A}}(-,X) = \mathbb{R}^n \mathrm{Hom}_{\mathcal{A}}(-,X)$$
$$\mathrm{Tor}^R_n(-,N) = \mathbb{L}_n(- \otimes_R N)$$
$$\mathrm{Tor}^R_n(M,-) = \mathbb{L}_n(M \otimes_R -)$$

Will see later: $\mathrm{Ext}^n_{\mathcal{A}}(X,-)(Y) \cong \mathrm{Ext}^n_{\mathcal{A}}(-,Y)(X)$

$\mathrm{Tor}^R_n(M,-)(N) \cong \mathrm{Tor}^R_n(-,N)(M)$

<u>Ex</u>: Compute $\mathrm{Tor}_i^{\mathbb{Z}}\left(\mathbb{Z}/n\mathbb{Z}, -\right)\left(\mathbb{Z}/m\mathbb{Z}\right)$

· $0 \to \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \to 0$ proj resolution of $\mathbb{Z}/m\mathbb{Z}$

· Apply $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} -$, to this, get

$$\dots 0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z}/n\mathbb{Z} \to 0 \dots$$

Now $\mathrm{Tor}_i^{\mathbb{Z}}\left(\mathbb{Z}/n\mathbb{Z}, -\right)\left(\mathbb{Z}/m\mathbb{Z}\right) = 0 \quad i \geq 2$

$$\mathrm{Tor}_1^{\mathbb{Z}}\left(\mathbb{Z}/n\mathbb{Z}, -\right)\left(\mathbb{Z}/m\mathbb{Z}\right) = \ker\left(\mathbb{Z}/n\mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z}/n\mathbb{Z}\right)$$

$$\cong \mathbb{Z}/\gcd(n,m)$$

Check that $\mathrm{Tor}_i^{\mathbb{Z}}\left(-, \mathbb{Z}/m\mathbb{Z}\right)\left(\mathbb{Z}/n\mathbb{Z}\right)$ gives the same result!

(2) Compute $\mathrm{Ext}_{\mathbb{Z}}^i\left(-, \mathbb{Z}/m\mathbb{Z}\right)\left(\mathbb{Z}/n\mathbb{Z}\right)$

degree -1 degree 0

$\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}$ proj resolution of $\mathbb{Z}/n\mathbb{Z}$

Apply $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/m\mathbb{Z})$:

$$\begin{tikzcd}
0 \arrow[r] & \overset{0}{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})} \arrow[r, "\cdot n"] \arrow[d, phantom, "\cong" rotate=90] & \overset{1}{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})} \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0 \dots \\
& \mathbb{Z}/m\mathbb{Z} \arrow[r, "\cdot n"] & \mathbb{Z}/m\mathbb{Z}
\end{tikzcd}$$

$$\mathrm{Ext}^i_{\mathbb{Z}}(-, \mathbb{Z}/m\mathbb{Z})(\mathbb{Z}/n\mathbb{Z}) = \begin{cases} 0 & i \geq 2 \\ \mathrm{Coker}(\mathbb{Z}/m\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}/m\mathbb{Z}) \end{cases} \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$$

Lecture 17

Syzygies and dimension shift

Def: $\mathcal{A}$ abelian, $A \in \mathcal{A}$.

- A syzygy of A, denoted ΩA, is the kernel of an epi $P \twoheadrightarrow A$ with P projective
- A cosyzygy of A, denoted $\mho A$ is the cokernel of a mono $A \rightarrowtail I$ with I injective.