repeating this procedure, we can construct maps $h^{-2}: P^{-2} \to Q^{-3}$, $h^{-3}: P^{-3} \to Q^{-4}$, ...

giving a null-homotopy of $f^{\bullet}$.

<u>Corollary</u>: Let $\mathcal{A}$ abelian cat with enough projectives. Then taking projective resolutions defines a functor

$$p: \mathcal{A} \to K(\mathcal{A})$$

such that $H^0 \circ p = id_{\mathcal{A}}$ and $H^n \circ p = 0$ $n \neq 0$.

<u>Proof</u>: This follows from the previous result.

Lecture 15

The dual result gives a functor $i: \mathcal{A} \to K(\mathcal{A})$ where iA - injective resolution of A, satisfying $H^0 \circ i = id_{\mathcal{A}}$ & $H^n \circ i = 0$ $\forall n \neq 0$.

<u>Lemma</u> (Horseshoe lemma) $\mathcal{A}$ abelian cat, and let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{A}$. Assume $P^{\cdot}$ and $Q^{\cdot}$ are projective resolutions of A and C respectively. Then there exists a proj resolution $R^{\cdot}$ of B with $R^n = P^n \oplus Q^n$ $\forall n$, s.t. the following diagram commutes

$$\begin{tikzcd}
\cdots P^{-2} \arrow[r] \arrow[d, tail, "\binom{1}{0}"] & P^{-1} \arrow[r] \arrow[d, tail, "\binom{1}{0}"] & P^{0} \arrow[r] \arrow[d, tail, "\binom{1}{0}"] & A \arrow[d, tail] \\
\cdots P^{-2}\oplus Q^{-2} \arrow[r] \arrow[d, two heads, "(0,1)"] & P^{-1}\oplus Q^{-1} \arrow[r] \arrow[d, two heads, "(0,1)"] & P^{0}\oplus Q^{0} \arrow[r] \arrow[d, two heads, "(0,1)"] & B \arrow[d, two heads] \\
\cdots Q^{-2} \arrow[r] & Q^{-1} \arrow[r] & Q^{0} \arrow[r] & C
\end{tikzcd}$$

<u>Proof</u>: Consider the diagram

$$\begin{tikzcd}
\operatorname{Ker} p_0 \arrow[r, tail] \arrow[d, dashed] & P^{0} \arrow[r, "p_0"] \arrow[d, "\binom{1}{0}"] & A \arrow[d, tail, "f"] \\
\operatorname{Ker} r_0 \arrow[r, dashed] \arrow[d, dashed] & P^{0}\oplus Q^{0} \arrow[r, dashed, "r_0"] \arrow[d, "(0,1)"] & B \arrow[d, two heads, "g"] \\
\operatorname{Ker} q_0 \arrow[r, tail] & Q^{0} \arrow[r, "q_0"] & C
\end{tikzcd}$$

Since $g: B \twoheadrightarrow C$ is an epi and Q^0 is proj
can find map $q_0': Q^0 \longrightarrow B$ s.t. $g \circ q_0' = q_0$
Set $r_0 := (f \circ p_0, q_0'): P^0 \oplus Q^0 \longrightarrow B$
Then get commutative diagram as indicated above. Now by the snake lemma the sequence
$0 \to \operatorname{Ker} p_0 \to \operatorname{Ker} r_0 \to \operatorname{Ker} q_0 \to 0$ is exact.

Hence we can repeat the procedure with
$0 \to A \to B \to C \to 0$ replaced by
$0 \to \operatorname{Ker} p_0 \to \operatorname{Ker} r_0 \to \operatorname{Ker} q_0 \to 0$.
This proves the claim ∎

Derived functors

$0 \to A \to B \to C \to 0$ exact, F right exact
$\Rightarrow \quad F(A) \to F(B) \to F(C) \to 0$ exact
Q: Can we continue the sequence on the left
Answer: Yes! Using derived functors

Definition and first properties

Let $F: \mathcal{A} \to \mathcal{B}$ be additive functor
$\Rightarrow F$ induce additive functor

$$F_{Ch}: Ch(\mathcal{A}) \longrightarrow Ch(\mathcal{B})$$

$$F_{Ch}\left(\dots \xrightarrow{d^{-2}} A^{-1} \xrightarrow{d^{-1}} A^{0} \xrightarrow{d^{0}} \dots \right)$$

$$= \left(\dots \xrightarrow{F(d^{-2})} F(A^{-1}) \xrightarrow{F(d^{-1})} F(A^{0}) \xrightarrow{F(d^{0})} \dots \right)$$

$f^{\bullet} = (f^n)_{n \in \mathbb{Z}}: A^{\bullet} \longrightarrow B^{\bullet}$ morphism in $Ch(\mathcal{A})$, then

$$F_{Ch}(f^{\bullet}) = (F(f^n))_{n \in \mathbb{Z}}: F_{Ch}(A^{\bullet}) \to F_{Ch}(B^{\bullet})$$

Can check that F_{Ch} preserves null-homotopic maps, so descends to a functor

$$F_K : K(\mathcal{A}) \to K(\mathcal{B})$$

s.t.

$$\begin{tikzcd} Ch(\mathcal{A}) \arrow[r, "F_{Ch}"] \arrow[d] & Ch(\mathcal{B}) \arrow[d] \\ K(\mathcal{A}) \arrow[r, "F_K"] & K(\mathcal{B}) \end{tikzcd}$$

commutes.

$\mathcal{A}$ abelian category. Recall that if

- $\mathcal{A}$ has enough proj's, then taking proj resolutions gives a functor $p : \mathcal{A} \to K(\mathcal{A})$
- $\mathcal{A}$ has enough inj's, then taking inj resolutions induce a functor $i : \mathcal{A} \to K(\mathcal{A})$

<u>Definition</u>: $F : \mathcal{A} \to \mathcal{B}$ additive functor between abelian categories

- If $\mathcal{A}$ has enough projectives & F is right exact, then the <u>nth left derived functor</u> of F is

$$\mathbb{L}_n F := H^{-n} \circ F_K \circ p : \mathcal{A} \to \mathcal{B}$$

- If $\mathcal{A}$ has enough injectives and F is left exact then the <u>n'th right derived functor</u> of F is

$$\mathbb{R}^n F = H^n \circ F \circ i : \mathcal{A} \longrightarrow \mathcal{B}$$

- $\mathbb{L}_n F(A)$ is computed as follows:

- Choose proj resolution of A

$$P^\bullet = \quad \cdots \longrightarrow P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0$$

- apply F pointwise

$$\cdots \longrightarrow F(P^{-2}) \xrightarrow{F(d^{-2})} F(P^{-1}) \xrightarrow{F(d^{-1})} F(P^0) \quad = F_{ch}(P_\bullet)$$

- Take homology in degree $-n$:

$$H^{-n} F_{ch}(P^\bullet)$$

<u>Lecture 16</u>

- $\mathbb{L}_n(f)$ for a morphism $f: A \to B$ is computed as follows:
  - Choose proj resolutions $P^\bullet$ and $Q^\bullet$ of A and B, and a lift $f^\bullet : P^\bullet \to Q^\bullet$ of f
  - Apply F pointwise to get
  $$g^\bullet = F_{ch}(f^\bullet) : F_{ch}(P^\bullet) \to F_{ch}(Q^\bullet)$$
  - Take homology in degree $-n$: $\mathbb{L}_n(f) = H^{-n}(g^\bullet)$