- $f^\bullet$ is a homotopy equivalence in $Ch(\mathcal{A})$ iff $f^\bullet$ is an isomorphism in $K(\mathcal{A})$

Lecture 14

Lemma: $\mathcal{A}$ abelian category. The following hold:

(1) If $f^\bullet : A^\bullet \to B^\bullet$ is nullhomotopic, then $H^n(f^\bullet) = 0 \ \forall n \in \mathbb{Z}$.

(2) H^n descends to a functor $H^n(-) : K(\mathcal{A}) \to \mathcal{A}$

$$\left(\begin{array}{ccc} Ch(\mathcal{A}) & \xrightarrow{H^n(-)} & \\ \downarrow & \circlearrowleft & Ab \\ K(\mathcal{A}) & \xrightarrow{H^n(-)} & \end{array} \right)$$

(3) If $f^\bullet$ is a homotopy equivalence, then $f^\bullet$ is a quasi-isomorphism.

Proof:

(1) (For Mod R): Have $f^n = d_B^{n-1} h^n + h^{n+1} d_A^n$ for morphisms $h^n : A^n \to B^{n-1}$. Hence for $x \in Z^n(A)$, have

$f^n(x) = d_B^{n-1} h^n(x) + h^{n+1} \underbrace{d_A^n(x)}_{0} = d_B^{n-1} h^n(x)$, so

$H^n(f)(x + \operatorname{im} d_A^{n-1}) = f(x) + \operatorname{im} d_B^{n-1} = d_B^{n-1}(h^n(x)) + \operatorname{im} d_B^{n-1} = 0$

so $H^n(f) = 0$

(2) If $f^\cdot \sim g^\cdot$, then $f^\cdot - g^\cdot$ is null-homotopic, so $0 = H^n(f^\cdot - g^\cdot) = H^n(f^\cdot) - H^n(g^\cdot)$. Hence, $H^n(f^\cdot) = H^n(g^\cdot)$. This implies the result.

(3) If $f^\cdot$ is a homotopy equivalence, then it is an isomorphism in $K(\mathcal{A})$.
Since $H^n(-)$ is a functor: $K(\mathcal{A}) \to \mathcal{A}$ it must preserve isomorphisms in $K(\mathcal{A})$.
Hence, if $f^\cdot$ is a homotopy equivalence, then $H^n(f)$ is an isomorphism $\forall n \in \mathbb{Z}$.
$\Rightarrow f^\cdot$ is a quasi-isomorphism ∎

Projective and injective resolutions

Def: $\mathcal{A}$ abelian cat.
(1) $\mathcal{A}$ has enough projectives if $\forall A \in \mathcal{A}$ there exists $P \in \mathcal{A}$ projective and an epimorphism $P \twoheadrightarrow \mathcal{A}$
(2) $\mathcal{A}$ has enough injectives if $\forall A \in \mathcal{A}$ there exists $I \in \mathcal{A}$ injective and a monomorphism $A \to I$

Ex: (1) Mod R has enough projectives and injectives

(2) mod $\mathbb{Z}$ – category of fin. gen abelian groups. Then mod $\mathbb{Z}$ has enough projectives, but not enough injectives

(3) There exist abelian categories with no injective or projective object, e.g. the category of coherent sheaves on the projective line $\mathbb{P}^1_{\mathbb{C}}$

Def: $\mathcal{A}$ abelian cat, and $A \in \mathcal{A}$.

(1) A projective resolution of A is a complex

$$P^\bullet: \cdots \to P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0 \to 0 \to 0 \to \cdots$$

with projective terms, which is exact everywhere except in position 0, where $H^0(P^\bullet) = \operatorname{Coker} d^{-1} \cong A$

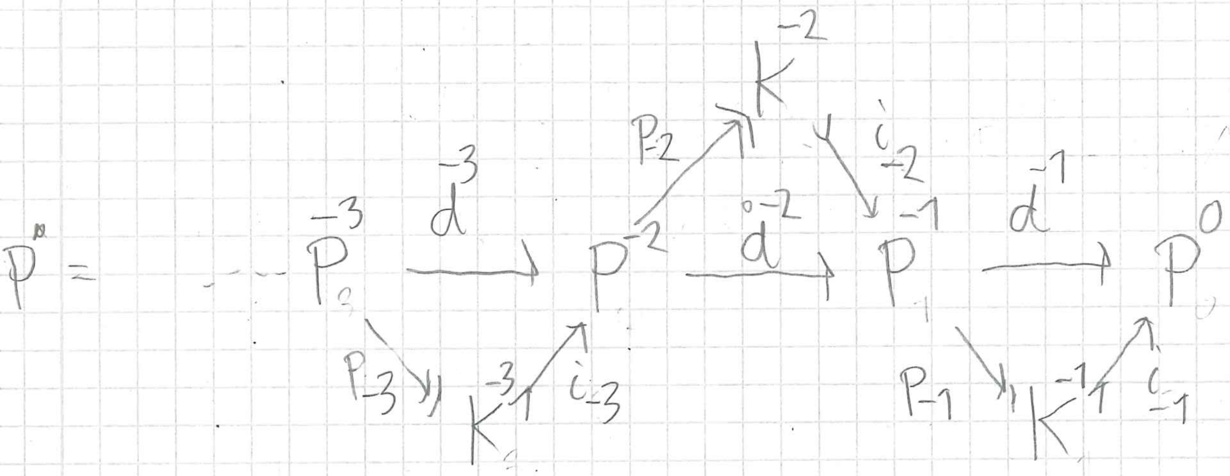
(2) An injective resolution of A is a complex

$$\cdots \to 0 \to 0 \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \to \cdots$$

with injective terms, which is exact everywhere except in position 0, where $H^0(I^\bullet) = \operatorname{Ker} d^0 \cong A$

Construction: Let $\mathcal{A}$ be an abelian cat with enough projectives. Then any $A \in \mathcal{A}$ has a projective resolution, constructed in the following way:

Let $0 \to K^{-1} \xrightarrow{i_{-1}} P^{0} \xrightarrow{p_0} A \to 0$ exact with P_0 projective



Iterative choose exact sequences

$0 \to K^{-n-1} \xrightarrow{i_{-n-1}} P^{-n} \xrightarrow{p_{-n}} K^{-n} \to 0$ with P^{-n} projective,

and set $d^{-n} = i_{-n} \circ p_{-n}$. Then $P^{\bullet}$ is a proj resolution of A.

If $\mathcal{A}$ has enough injectives, the dual construction works to get an injective coresolution of A.

Lemma: $\mathcal{A}$ abelian cat with enough projectives.

(1) Any object $A \in \mathcal{A}$ has a projective resolution

(2) Let $f: A \to B$ morphism in $\mathcal{A}$, and let $P^\bullet$ and $Q^\bullet$ be projective resolutions of A and B, respectively. Then f can be lifted to a morphism $f^\bullet: P^\bullet \to Q^\bullet$ of chain complexes, i.e. there exists morphisms $f^{-n}: P^{-n} \to Q^{-n}$ s.t.

$$\begin{array}{ccccccccc}
\to & P^{-2} & \xrightarrow{d_P^{-2}} & P^{-1} & \xrightarrow{d_P^{-1}} & P^{0} & \longrightarrow & A & \to 0 \\
 & \downarrow{\scriptstyle f^{-2}} & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^{0}} & & \downarrow{\scriptstyle f} & \\
\to & Q^{-2} & \xrightarrow{d_Q^{-2}} & Q^{-1} & \xrightarrow{d_Q^{-1}} & Q^{0} & \longrightarrow & B & \to 0
\end{array}$$

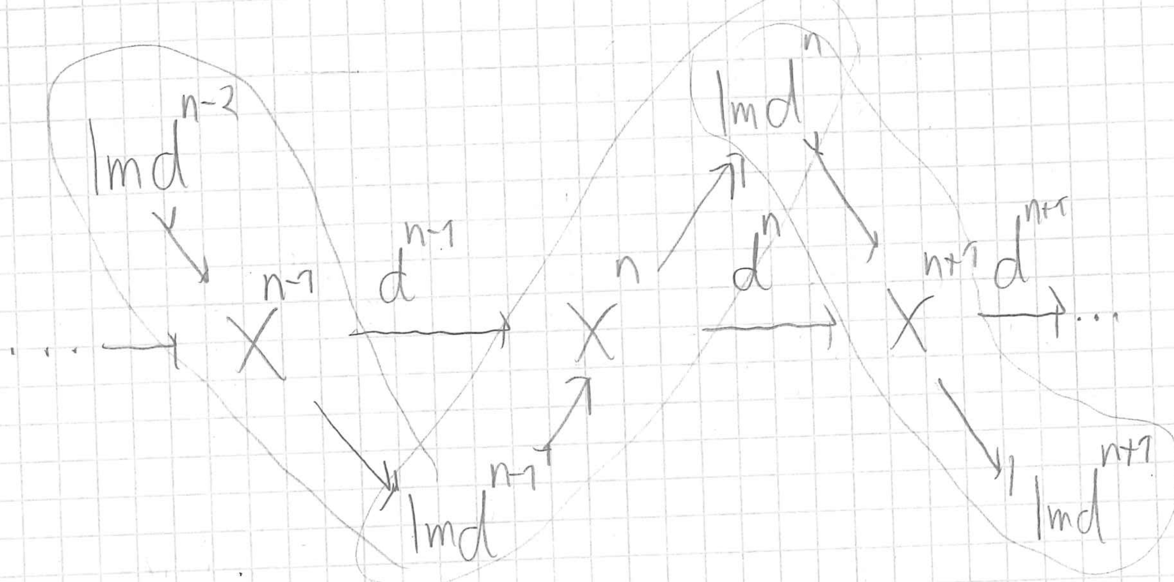
commutes.

(3) The lifting in (2) is unique up to homotopy, i.e. if $\tilde{f}^\bullet: P^\bullet \to Q^\bullet$ is another lift of f, then $\tilde{f}^\bullet \sim f^\bullet$

Proof

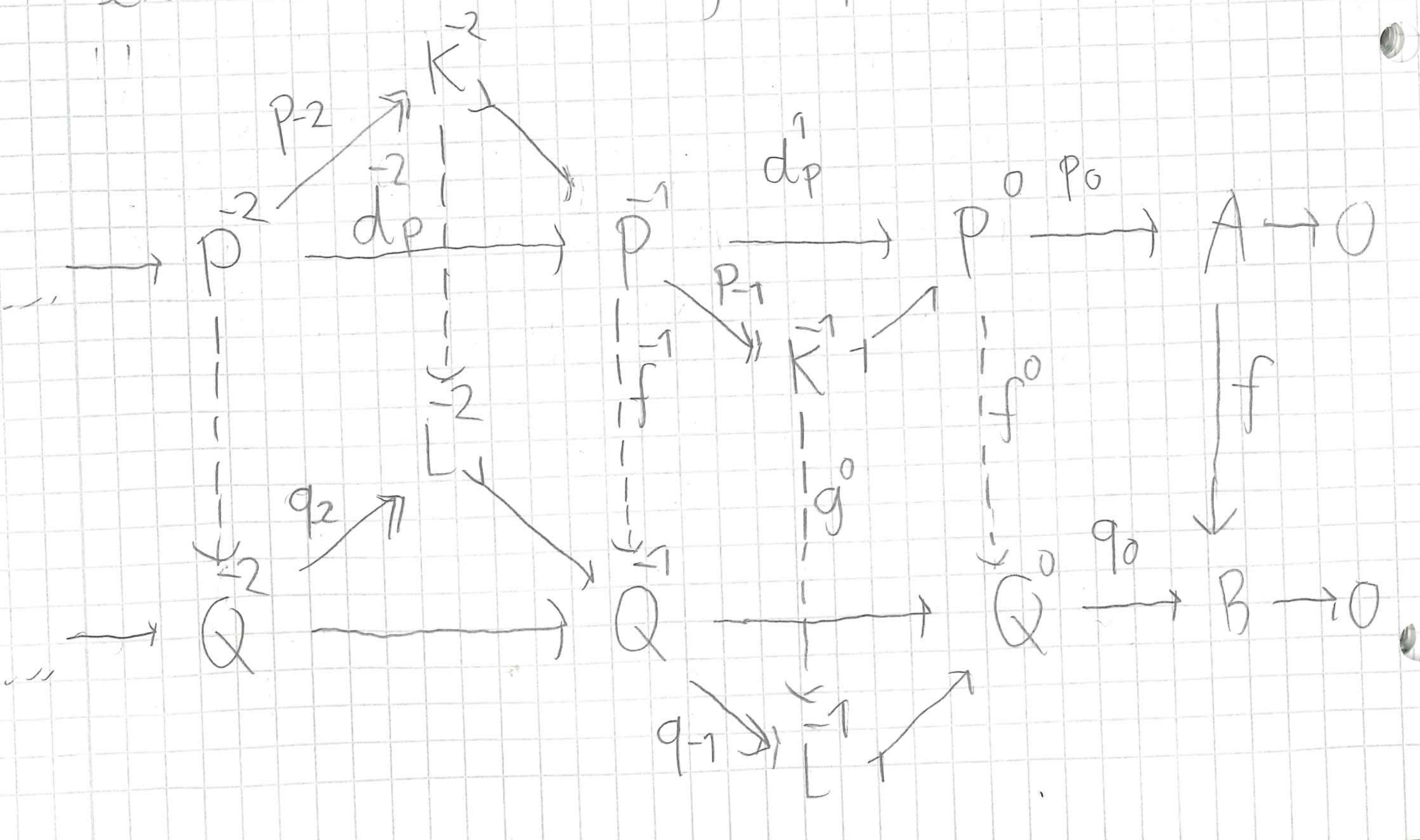
(1) Follows from the construction above.

(2) Note that any exact sequence $X^\bullet$ can be constructed via combining short exact sequences, since $\operatorname{im} d^{n-1} = \operatorname{Ker} d^n$.



Here $0 \to \operatorname{Im} d^{i-1} \rightarrowtail X^n \twoheadrightarrow \operatorname{Im} d^i \to 0$

exact $\forall i \in \mathbb{Z}$. Using this, consider



f^0 exists, since P^0 projective & q_0 epi.

g^0 exists by commutativity of the right hand square.

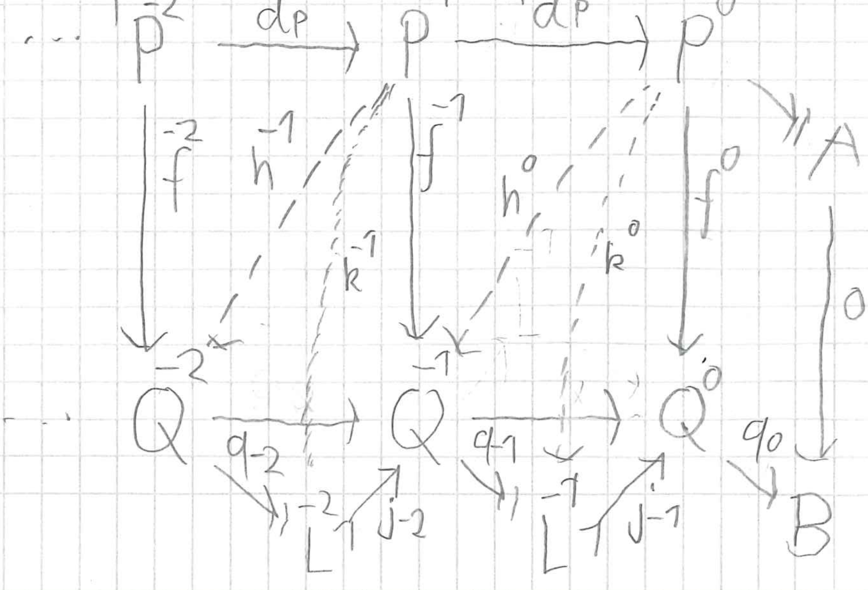
f^{-1} is a lift of the map $g^0 \circ p_{-1}$, which exists since P^{-1} projective & q_{-1} epi

g^{-1} exists by commutativity of square with maps $p_{-1}, g^0, q_{-1}, f^{-1}$.

⋮

repeating this procedure, we get maps $f^{-n}: P^{-n} \to Q^{-n}$ which forms a morphism $f^\bullet: P^\bullet \to Q^\bullet$ in $Ch(\mathcal{A})$ lifting f

To prove (3) it suffices to show that $f^\bullet - \tilde{f}^\bullet$ is null-homotopic, i.e. that any lift of the zero map is nullhomotopic. So consider



Since $q_0 \circ f^0 = 0$, the map f^0 factors through the kernel L^{-1} of q_0 via a map $k^0 : P^0 \to Q^{-1}$.
Since $q_{-1} : Q \to L$ is an epimorphism & P^0 is projective, we can find lift $h^0 : P^0 \to Q^{-1}$ satisfying $q_{-1} \circ h^0 = k^0$.

Next consider the difference

$$f^{-1} - h^0 \circ d_P^{-1} : P^{-1} \to Q^{-1}.$$

Since $q_{-1} \circ (f^{-1} - h^0 \circ d_P^{-1}) = q_{-1} \circ f^{-1} - k^0 \circ d_P^{-1}$

and $j_{-1} \circ (q_{-1} \circ f^{-1} - k^0 \circ d_P^{-1})$

$$= (j_{-1} \circ q_{-1}) \circ f^{-1} - (j_{-1} \circ k^0) \circ d_P^{-1} = d_Q^{-1} \circ f^{-1} - f^0 \circ d_P^{-1} = 0$$

and j_{-1} is a monomorphism, it follows that $q_{-1} \circ (f^{-1} - h^0 \circ d_P^{-1}) = 0$, so $f^{-1} - h^0 \circ d_P^{-1}$ factors through L^{-2} via a morphism $k^{-1} : P^{-1} \to L^{-2}$.
Again, since P^{-1} is projective & $q_{-2} : Q^{-2} \to L^{-2}$ is an epimorphism, we can find a morphism $h^{-1} : P^{-1} \to Q^{-2}$ s.t.

$$q_{-2} \circ h^{-1} = k^{-1}$$

It follows that $d_Q^{-2} \circ h^{-1} = f^{-1} - h^0 \circ d_P^{-1}$

$$\Leftrightarrow f^{-1} = d_Q^{-2} \circ h^{-1} + h^0 \circ d_P^{-1}$$

repeating this procedure, we can construct maps $h^{-2}: P^{-2} \to Q^{-3}$, $h^{-3}: P^{-3} \to Q^{-4}$, ...

giving a null-homotopy of $f^\bullet$.

Corollary: Let $\mathcal{A}$ abelian cat with enough projectives. Then taking projective resolutions defines a functor

$$p: \mathcal{A} \to K(\mathcal{A})$$

such that $H^0 \circ p = id_{\mathcal{A}}$ and $H^n \circ p = 0$ $n \neq 0$.

Proof: This follows from the previous result.

Lecture 15

The dual result gives a functor $i: \mathcal{A} \to K(\mathcal{A})$ where iA - injective resolution of A, satisfying $H^0 \circ i = id_{\mathcal{A}}$ & $H^n \circ i = 0$ $\forall n \neq 0$.

Lemma (Horseshoe lemma) $\mathcal{A}$ abelian cat, and let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{A}$. Assume $P^\bullet$ and $Q^\bullet$ are projective resolutions of A and C respectively. Then there exists a proj resolution $R^\bullet$ of B with $R^n = P^n \oplus Q^n$ $\forall n$, s.t. the following diagram commutes