

Lecture 23

Lemma. (Ore condition II) Let $f, g: A^* \rightarrow B^*$ in $\mathcal{K}(A)$. Then the following are equivalent.

(i) There exists a quasi isomorphism $s: B^* \rightarrow S^*$ satisfying $s \circ f = s \circ g$.

(ii) There exists a quasi isomorphism $t: T^* \rightarrow A^*$ satisfying $f \circ t = g \circ t$.

Proof. We only show (ii) \Rightarrow (i); the other direction is similar. Since $f \circ t = g \circ t$, we have $(f-g) \circ t = 0$. We apply (TR3) to

$$\begin{array}{ccccccc} T & \xrightarrow{t} & A & \xrightarrow{\text{Cone}(t)} & T[C_1] \\ 0 \downarrow & & f-g \downarrow & & \downarrow \\ 0 & \longrightarrow & B & \xrightarrow{\text{id}_B} & B & \longrightarrow & 0 \end{array}$$

to obtain a commutative diagram

$$\begin{array}{ccccccc} T & \xrightarrow{t} & A & \xrightarrow{\text{Cone}(t)} & T[C_1] \\ 0 \downarrow & & f-g \downarrow & & \downarrow h \\ 0 & \longrightarrow & B & \xrightarrow{\text{id}_B} & B & \longrightarrow & 0 \end{array}$$

By applying (TR1)(iii) to h we obtain a diagram

$$\begin{array}{ccccccc}
T & \xrightarrow{t} & A & \xrightarrow{d} & \text{Cone}(t) & \longrightarrow & T[\mathbb{I}] \\
0 \downarrow & & F-g \downarrow & & \downarrow h & & \downarrow \\
0 & \longrightarrow & B & \xrightarrow{\text{id}_B} & B & \longrightarrow & 0 \\
& & & & \downarrow s & & \\
& & & & \text{Cone}(h) & & \\
& & & & \downarrow & & \\
& & & & \text{Cone}(t)[\mathbb{I}] & &
\end{array}$$

Then

$$0 \circ d = s \circ h \circ d = s \circ (F-g) \Rightarrow s \circ f = s \circ g.$$

It remains to show that s is a quasi isomorphism. But we have

$$\text{Cone}(s) \cong \text{Cone}(t)[\mathbb{I}]$$

and

t quasi isomorphism $\Leftrightarrow \text{Cone}(t)$ exact

$\Leftrightarrow \text{Cone}(t)[\mathbb{I}]$ exact

$\Leftrightarrow \text{Cone}(s)$ exact

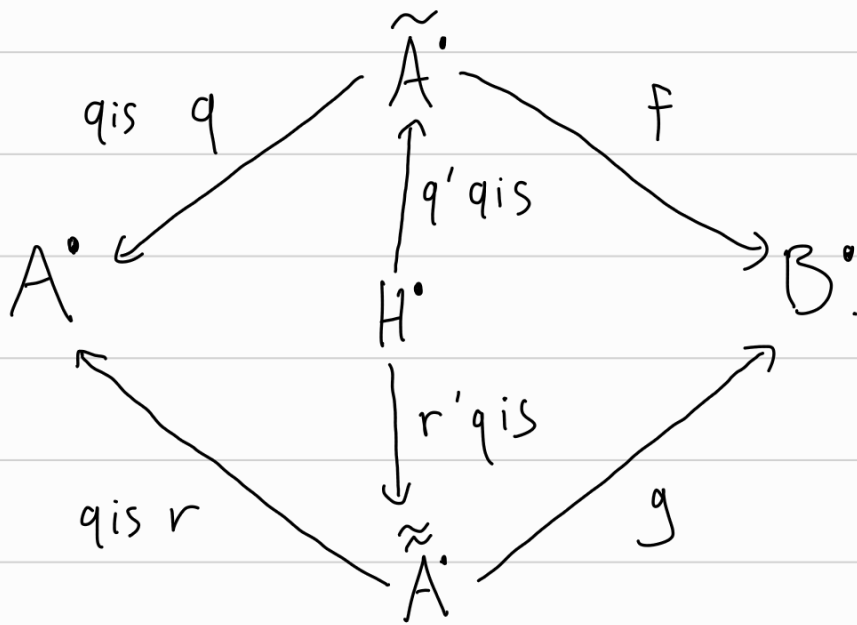
$\Leftrightarrow s$ quasi isomorphism. \square

Definition (1) Let $A^\bullet, B^\bullet \in \mathcal{C}(A)$. A roof from A^\bullet to B^\bullet is a diagram of the form

$$\begin{array}{ccc}
& \tilde{A}^\bullet & \\
q \text{ is } q \swarrow & & \searrow f \\
A^\bullet & & B^\bullet
\end{array}$$

where $\tilde{A} \in \mathcal{L}(A)$ and q is a quasi-isomorphism. We denote such a roof by $f \cdot q^{-1}$.

(ii) Two roofs $f \cdot q^{-1}$ and $g \cdot r^{-1}$ are called equivalent if there exist a commutative diagram

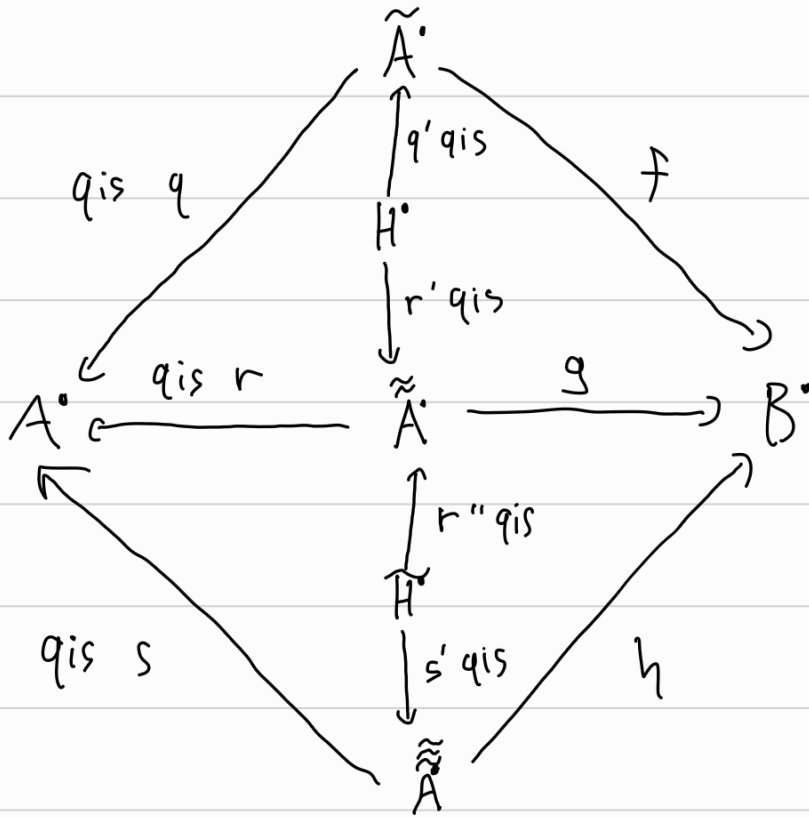


Lemma. Equivalence of roofs is an equivalence relation.

Proof. Let $f \cdot q^{-1}$, $g \cdot r^{-1}$, $h \cdot s^{-1}$ be three roofs. Clearly $f \cdot q^{-1}$ is equivalent to $f \cdot q^{-1}$ (reflexive). Clearly if $f \cdot q^{-1}$ is equivalent to $g \cdot r^{-1}$, then $g \cdot r^{-1}$ is equivalent to $f \cdot q^{-1}$ (symmetric).

Assume that $f \cdot q^{-1}$ is equivalent to $g \cdot r^{-1}$ and that $g \cdot r^{-1}$ is equivalent to $h \cdot s^{-1}$. We

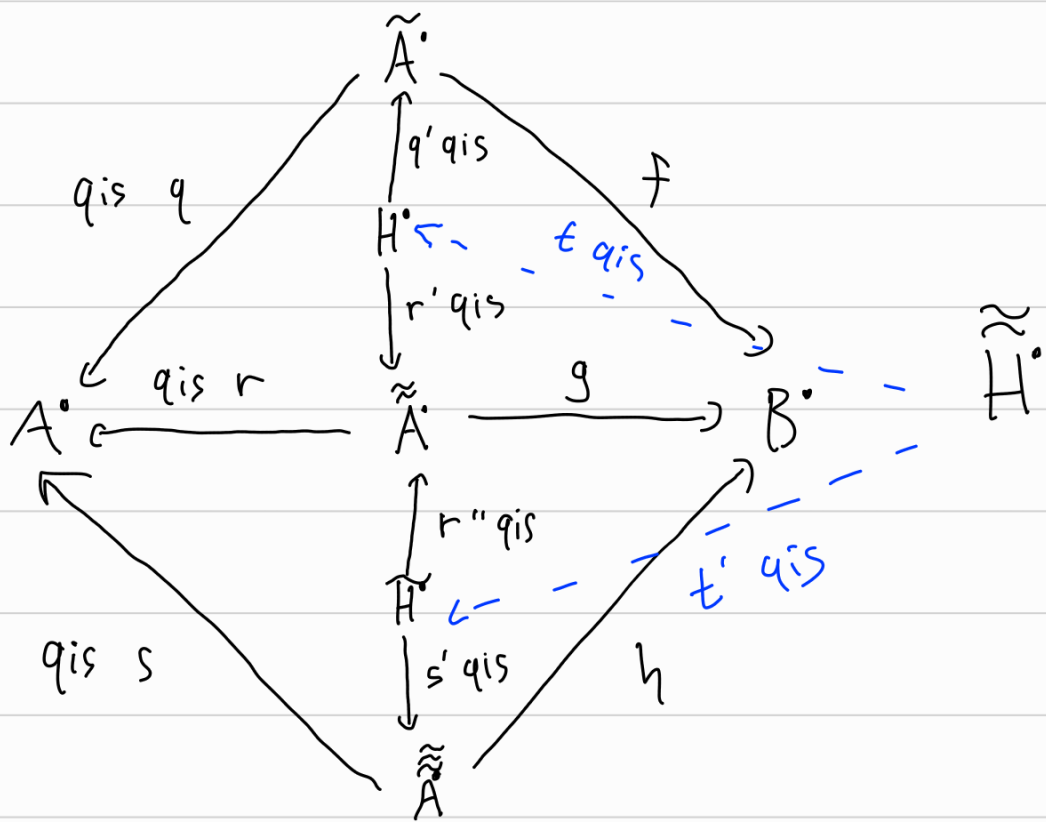
show that $f \cdot q'$ is equivalent to $h \cdot s'$ (transitive).
 Then we have the following commutative diagram:



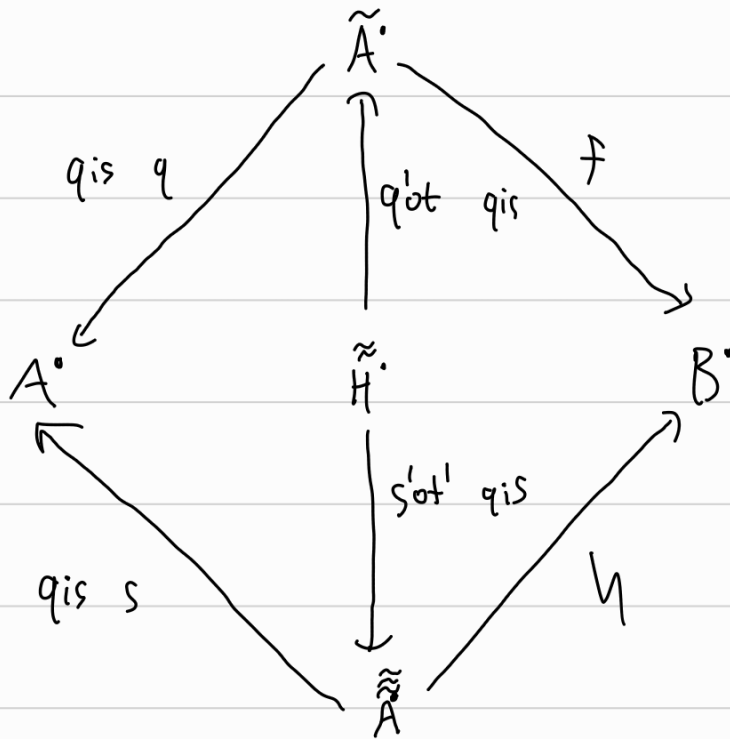
Applying Ore condition I to

$$\begin{array}{ccc} & H' & \\ & \downarrow qisr' & \\ \tilde{H}' & \xrightarrow{qisr''} & \tilde{\tilde{A}} \end{array}, \quad r'ot = r''ot'$$

we obtain a commutative diagram



where if we specialize to



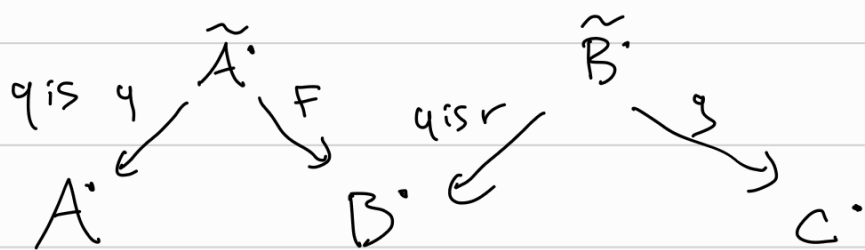
we obtain that $f \cdot q^{-1}$ is equivalent to $h \cdot s^{-1}$. \square

Definition. Assume that for any $A, B \in \mathcal{C}(\mathcal{A})$, the collection of roofs from A to B up to equivalence

nce is a set. The derived category of \mathcal{A} , $\mathcal{D}(\mathcal{A})$ is defined by

$$\text{Ob}(\mathcal{D}(\mathcal{A})) = \text{Ob}(\mathcal{K}(\mathcal{A}))$$

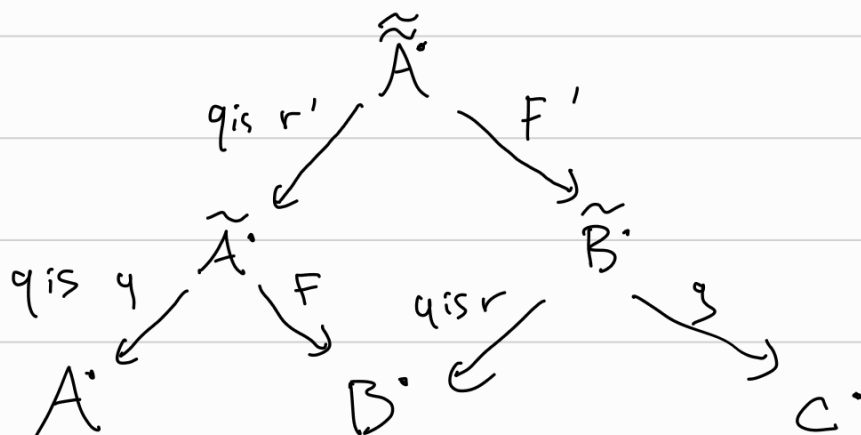
Hom $_{\mathcal{D}(\mathcal{A})}(A^\bullet, B^\bullet) = \{ \text{roofs from } A^\bullet \text{ to } B^\bullet \} / \sim$
 if $f \cdot q^{-1}: A^\bullet \rightarrow B^\bullet$ and $g \cdot r^{-1}: B^\bullet \rightarrow C^\bullet$, then



and applying One Condition I to



we obtain a commutative diagram



from which we define

$$(g \cdot r^{-1}) \circ (F \cdot q^{-1}) := (g \circ F') \cdot (q \circ r')^{-1}$$