Lecture 23

<u>Lemma</u>. (Ore condition II) Let $f, g : A^\bullet \to B^\bullet$ in $K(\mathcal{A})$.
Then the following are equivalent.
(i) There exists a quasi isomorphism $s : B^\bullet \to S^\bullet$ satisfying $s \circ f = s \circ g$.
(ii) There exists a quasi isomorphism $t : T^\bullet \to A^\bullet$ satisfying $f \circ t = g \circ t$.

<u>Proof</u>. We only show (ii) $\Rightarrow$ (i); the other direction is similar. Since $f \circ t = g \circ t$, we have $(f-g) \circ t = 0$.
We apply (TR3) to

$$\begin{tikzcd}
T \arrow[r, "t"] \arrow[d, "0"] & A \arrow[r] \arrow[d, "f-g"] & \mathrm{Cone}(t) \arrow[r] & T[1] \arrow[d] \\
0 \arrow[r] & B \arrow[r, "id_B"] & B \arrow[r] & 0
\end{tikzcd}$$

to obtain a commutative diagram

$$\begin{tikzcd}
T \arrow[r, "t"] \arrow[d, "0"] & A \arrow[r] \arrow[d, "f-g"] & \mathrm{Cone}(t) \arrow[r] \arrow[d, "h"] & T[1] \arrow[d] \\
0 \arrow[r] & B \arrow[r, "id_B"] & B \arrow[r] & 0
\end{tikzcd}$$

By applying (TR1)(iii) to h we obtain a diagram

$$\begin{tikzcd}
T \arrow[r, "t"] \arrow[d, "0"'] & A \arrow[r, "d"] \arrow[d, "f-g"'] & \mathrm{Cone}(t) \arrow[r] \arrow[d, "h"] & T[1] \arrow[d] \\
0 \arrow[r] & B \arrow[r, "\mathrm{id}_B"] & B \arrow[r] \arrow[d, "s"] & 0 \\
& & \mathrm{Cone}(h) \arrow[d] & \\
& & \mathrm{Cone}(t)[1] &
\end{tikzcd}$$

Then

$$0 \circ d = s \circ h \circ d = s \circ (f-g) \implies s \circ f = s \circ g.$$

It remains to show that s is a quasi isomorphism. But we have

$$\mathrm{Cone}(s) \cong \mathrm{Cone}(t)[1]$$

and

$$\begin{aligned}
t \text{ quasi isomorphism} &\iff \mathrm{Cone}(t) \text{ exact} \\
&\iff \mathrm{Cone}(t)[1] \text{ exact} \\
&\iff \mathrm{Cone}(s) \text{ exact} \\
&\iff s \text{ quasi isomorphism.}
\end{aligned}$$

$\square$

<u>Definition</u> (i) Let $A^\bullet, B^\bullet \in \mathcal{C}(\mathcal{A})$. A <u>roof from $A^\bullet$ to $B^\bullet$</u> is a diagram of the form

$$\begin{tikzcd}
& \tilde{A}^\bullet \arrow[dl, "\text{qis } q"'] \arrow[dr, "f"] & \\
A^\bullet & & B^\bullet
\end{tikzcd}$$

where $\tilde{A}^\bullet \in \mathcal{C}(\mathcal{A})$ and q is a quasi-isomorphism. We denote such a roof by $f \cdot q^{-1}$.

(ii) Two roofs $f \cdot q^{-1}$ and $g \cdot r^{-1}$ are called <u>equivalent</u> if there exist a commutative diagram

$$\begin{tikzcd}
& \tilde{A}^\bullet \arrow[dl, "\text{qis } q"'] \arrow[dr, "f"] & \\
A^\bullet & H^\bullet \arrow[u, "q' \text{ qis}"'] \arrow[d, "r' \text{ qis}"] & B^\bullet \\
& \tilde{\tilde{A}}^\bullet \arrow[ul, "\text{qis } r"] \arrow[ur, "g"'] &
\end{tikzcd}$$

<u>Lemma</u>. Equivalence of roofs is an equivalence relation.

<u>Proof</u>. Let $f \cdot q^{-1}$, $g \cdot r^{-1}$, $h \cdot s^{-1}$ be three roofs. Clearly $f \cdot q^{-1}$ is equivalent to $f \cdot q^{-1}$ (reflexive). Clearly if $f \cdot q^{-1}$ is equivalent to $g \cdot r^{-1}$, then $g \cdot r^{-1}$ is equivalent to $f \cdot q^{-1}$ (symmetric). Assume that $f \cdot q^{-1}$ is equivalent to $g \cdot r^{-1}$ and that $g \cdot r^{-1}$ is equivalent to $h \cdot s^{-1}$. We

show that $f \cdot q^{-1}$ is equivalent to $h \cdot s^{-1}$ (transitive).
Then we have the following commutative diagram:

$$
\begin{tikzcd}
 & \tilde{A}^\bullet \arrow[ddl, "qis\ q"'] \arrow[ddr, "f"] & \\
 & H^\bullet \arrow[u, "q'\ qis"'] \arrow[d, "r'\ qis"] & \\
A^\bullet & \tilde{\tilde{A}}^\bullet \arrow[l, "qis\ r"'] \arrow[r, "g"] & B^\bullet \\
 & \tilde{H}^\bullet \arrow[u, "r''\ qis"'] \arrow[d, "s'\ qis"] & \\
 & \tilde{\tilde{\tilde{A}}}^\bullet \arrow[uul, "qis\ s"] \arrow[uur, "h"'] &
\end{tikzcd}
$$

Applying Ore condition I to

$$
\begin{tikzcd}
 & H^\bullet \arrow[d, "qis\ r'"] \\
\tilde{H}^\bullet \arrow[r, "qis\ r''"] & \tilde{\tilde{A}}^\bullet
\end{tikzcd}
,\qquad r' \circ t = r'' \circ t'
$$

we obtain a commutative diagram

$$
\begin{tikzcd}
& \tilde{A}^\bullet \arrow[ddl, "\text{qis}\ q"'] \arrow[ddr, "f"] & & \\
& H^\bullet \arrow[u, "q'\ \text{qis}"'] \arrow[d, "r'\ \text{qis}"] & & \\
A^\bullet & \tilde{\tilde{A}}^\bullet \arrow[l, "\text{qis}\ r"'] \arrow[r, "g"] & B^\bullet & \tilde{\tilde{H}}^\bullet \arrow[llu, dashed, "t\ \text{qis}"'] \arrow[lld, dashed, "t'\ \text{qis}"] \\
& \tilde{H}^\bullet \arrow[u, "r''\ \text{qis}"'] \arrow[d, "s'\ \text{qis}"] & & \\
& \tilde{\tilde{\tilde{A}}}^\bullet \arrow[uul, "\text{qis}\ s"] \arrow[uur, "h"'] & &
\end{tikzcd}
$$

where if we specialize to

$$
\begin{tikzcd}
& \tilde{A}^\bullet \arrow[dl, "\text{qis}\ q"'] \arrow[dr, "f"] & \\
A^\bullet & \tilde{\tilde{H}}^\bullet \arrow[u, "q'\circ t\ \ \text{qis}"'] \arrow[d, "s'\circ t'\ \ \text{qis}"] & B^\bullet \\
& \tilde{\tilde{\tilde{A}}}^\bullet \arrow[ul, "\text{qis}\ s"] \arrow[ur, "h"'] &
\end{tikzcd}
$$

we obtain that $f\cdot q^{-1}$ is equivalent to $h\cdot s^{-1}$. □

Definition. Assume that for any $A^\bullet, B^\bullet \in \mathcal{C}(\mathcal{A})$, the collection of roofs from $A^\bullet$ to $B^\bullet$ up to equivalence is a set. The <u>derived category of $\mathcal{A}$</u>, $\mathcal{D}(\mathcal{A})$ is defined by

$$Ob(\mathcal{D}(\mathcal{A})) = Ob(\mathcal{K}(\mathcal{A}))$$

$$Hom_{\mathcal{D}(\mathcal{A})}(A^\bullet, B^\bullet) = \{\text{roofs from } A^\bullet \text{ to } B^\bullet\}/\sim$$

if $f\cdot q^{-1}: A^\bullet \to B^\bullet$ and $g\cdot r^{-1}: B^\bullet \to C^\bullet$, then

$$\begin{tikzcd}
 & \tilde{A}^\bullet \arrow[dl, "q \text{ is } q"'] \arrow[dr, "f"] & & \tilde{B}^\bullet \arrow[dl, "q\text{is } r"'] \arrow[dr, "g"] & \\
A^\bullet & & B^\bullet & & C^\bullet
\end{tikzcd}$$

and applying Ore Condition I to

$$\begin{tikzcd}
 & \tilde{B}^\bullet \arrow[d, "r \text{ qis}"] \\
\tilde{A}^\bullet \arrow[r, "f"'] & B^\bullet
\end{tikzcd}$$

we obtain a commutative diagram

$$\begin{tikzcd}
 & & \tilde{\tilde{A}}^\bullet \arrow[dl, "q\text{is } r'"'] \arrow[dr, "f'"] & & \\
 & \tilde{A}^\bullet \arrow[dl, "q \text{ is } q"'] \arrow[dr, "f"] & & \tilde{B}^\bullet \arrow[dl, "q\text{is } r"'] \arrow[dr, "g"] & \\
A^\bullet & & B^\bullet & & C^\bullet
\end{tikzcd}$$

from which we define

$$(g\cdot r^{-1})\circ(f\cdot q^{-1}) := (g\circ f')\cdot(q\circ r')^{-1}.$$