Lecture 22

Theorem. $(\mathcal{K}(\mathcal{A}), [1], \Delta)$ is a triangulated category.

Proof. (TR1) – (TR3) were proved in previous lecture.

(TR4): Since we have (TR1), we may assume that triangles are standard triangles. Hence we assume that we have

$$\begin{tikzcd}
A^\bullet \arrow[r, "f^\bullet"] \arrow[d, equal] & B^\bullet \arrow[r, "{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}"] \arrow[d, "g^\bullet"] & \mathrm{Cone}(f^\bullet) \arrow[r, "{(0\ 1)}"] & A^\bullet[1] \arrow[d, equal] \\
A^\bullet \arrow[r, "g^\bullet \circ f^\bullet"] & C^\bullet \arrow[r, "{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}"] \arrow[d, "{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}"] & \mathrm{Cone}(g^\bullet \circ f^\bullet) \arrow[r, "{(0\ 1)}"] & A^\bullet[1] \arrow[d, "{f^\bullet[1]}"] \\
& \mathrm{Cone}(g^\bullet) \arrow[r, equal] \arrow[d, "{(0\ 1)}"] & \mathrm{Cone}(g^\bullet) \arrow[r, "{(0\ 1)}"] & B^\bullet[1] \\
& B^\bullet[1] \arrow[r, "{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}"] & \mathrm{Cone}(f^\bullet)[1] &
\end{tikzcd}$$

We introduce the morphisms

$$
\begin{tikzcd}[column sep=large, row sep=large]
A^\bullet \arrow[r, "f^\bullet"] \arrow[d, equal] & B^\bullet \arrow[r, "\begin{pmatrix}1\\0\end{pmatrix}"] \arrow[d, "g^\bullet"] & \mathrm{Cone}(f^\bullet) \arrow[r, "\begin{pmatrix}0 & 1\end{pmatrix}"] \arrow[d, dashed, "\begin{pmatrix}g^\bullet & 0\\0 & 1\end{pmatrix}"] & A^\bullet[1] \arrow[d, equal] \\
A^\bullet \arrow[r, "g^\bullet\circ f^\bullet"] & C^\bullet \arrow[r, "\begin{pmatrix}1\\0\end{pmatrix}"] \arrow[d, "\begin{pmatrix}1\\0\end{pmatrix}"] & \mathrm{Cone}(g^\bullet\circ f^\bullet) \arrow[r, "\begin{pmatrix}0 & 1\end{pmatrix}"] \arrow[d, dashed, "\begin{pmatrix}1 & 0\\0 & f^\bullet[1]\end{pmatrix}"] & A^\bullet[1] \arrow[d, "f^\bullet[1]"] \\
 & \mathrm{Cone}(g^\bullet) \arrow[r, equal] \arrow[d, "\begin{pmatrix}0 & 1\end{pmatrix}"] & \mathrm{Cone}(g^\bullet) \arrow[r, "\begin{pmatrix}0 & 1\end{pmatrix}"] \arrow[d, dashed, "\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}"] & B^\bullet[1] \\
 & B^\bullet[1] \arrow[r, "\begin{pmatrix}1\\0\end{pmatrix}"] & \mathrm{Cone}(f^\bullet)[1] &
\end{tikzcd}
$$

and we see that every square in the above diagram commutes (in $\mathcal{C}(\mathcal{A})$ even). It remains to show that the third column is isomorphic to a standard triangle in $\mathcal{K}(\mathcal{A})$, which is left as an exercise. $\square$

Derived functors

Let $A \in \mathcal{A}$ be an object. We identify it with the complex $\cdots \to 0 \to A \to 0 \to \cdots$ in $\mathcal{C}(\mathcal{A})$ where A appears in degree 0. Hence we have embeddings

$$\mathcal{A} \hookrightarrow \mathcal{C}(\mathcal{A}), \quad \mathcal{A} \hookrightarrow \mathcal{K}(\mathcal{A}).$$

<u>Lemma</u>. Let $X \in \mathcal{A}$ and $A^\bullet \in \mathcal{C}(\mathcal{A})$. Consider the complex $\mathrm{Hom}_{\mathcal{A}}(X, A^\bullet)$, that is

$$\cdots \to \mathrm{Hom}_{\mathcal{A}}(X, A^{-1}) \xrightarrow{d_A^{-1}\circ -} \mathrm{Hom}_{\mathcal{A}}(X, A^{0}) \xrightarrow{d_A^{0}\circ -} \mathrm{Hom}_{\mathcal{A}}(X, A^{1}) \longrightarrow \cdots$$

Then

$$Z^n(\mathrm{Hom}_{\mathcal{A}}(X, A^\bullet)) = \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X, A^\bullet[n]), \text{ and}$$
$$H^n(\mathrm{Hom}_{\mathcal{A}}(X, A^\bullet)) = \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X, A^\bullet[n]).$$

Dually

$$Z^n(\mathrm{Hom}_{\mathcal{A}}(A^\bullet, X)) = \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(A^\bullet, X[n]), \text{ and}$$
$$H^n(\mathrm{Hom}_{\mathcal{A}}(A^\bullet, X)) = \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A^\bullet, X[n]).$$

<u>Proof</u>. We only show the first two claims. We have

$$\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X, A^\bullet[n]) = \{\varphi : X \to A^n \mid d_A^n \circ \varphi = 0\}$$
$$= \mathrm{Ker}(d_A^n \circ - : \mathrm{Hom}_{\mathcal{A}}(X, A^n) \to \mathrm{Hom}_{\mathcal{A}}(X, A^{n+1})).$$
$$= Z^n(\mathrm{Hom}_{\mathcal{A}}(A^\bullet, X)).$$

To compute $H^n(\mathrm{Hom}_{\mathcal{A}}(A^\bullet, X))$ we need to compute $B^n(\mathrm{Hom}_{\mathcal{A}}(A^\bullet, X))$. We have

$$
\begin{aligned}
B^n(\mathrm{Hom}_{\mathcal{A}}(X, A^\bullet)) &= \mathrm{Im}\left(d^{n-1}\circ - : \mathrm{Hom}_{\mathcal{A}}(X, A^{n-1}) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, A^n)\right) \\
&= \{\varphi : X \longrightarrow A^n \mid \exists h : X \longrightarrow A^{n-1} \text{ with } \varphi = d^{n-1}\circ h\} \\
&= \{\varphi \in \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X, A^\bullet[n]) \mid \varphi \text{ null-homotopic}\}
\end{aligned}
$$

Hence

$$
\begin{aligned}
H^n(\mathrm{Hom}_{\mathcal{A}}(X, A^\bullet)) &= \frac{Z^n(\mathrm{Hom}_{\mathcal{A}}(X, A^\bullet))}{B^n(\mathrm{Hom}_{\mathcal{A}}(X, A^\bullet))} \\
&= \frac{\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X, A^\bullet[n])}{\{\text{null-homotopic maps in } \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X, A^\bullet[n])\}} \\
&= \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X, A^\bullet[n]).
\end{aligned}
$$

$\square$

Now assume that $\mathcal{A}$ is abelian with enough projectives. Recall the functor $p : \mathcal{A} \to \mathcal{K}(\mathcal{A})$ that maps an object to its projective resolution. By the horseshoe lemma (lecture 15) a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \quad \text{in } \mathcal{A}$$

gives rise to a triangle

$$p(A) \longrightarrow p(B) \longrightarrow p(C) \longrightarrow p(A)[1] \quad \text{in } \mathcal{K}(\mathcal{A}).$$

Indeed:

$$
\begin{tikzcd}
p(A) = \cdots \arrow[r] & P^{-2} \arrow[r, "f^{-2}"] \arrow[d, "\binom{1}{0}"] & P^{-1} \arrow[r, "f^{-1}"] \arrow[d, "\binom{1}{0}"] & P^{0} \arrow[r, "0"] \arrow[d, "\binom{1}{0}"] & 0 \arrow[r] \arrow[d, "0"] & \cdots \\
p(B) = \cdots \arrow[r] & P^{-2} \oplus Q^{-2} \arrow[r, "\begin{pmatrix} f^{-2} & h^{-2} \\ 0 & g^{-2} \end{pmatrix}"] \arrow[d, "(0\ 1)"] & P^{-1} \oplus Q^{-1} \arrow[r, "\begin{pmatrix} f^{-1} & h^{-1} \\ 0 & g^{-1} \end{pmatrix}"] \arrow[d, "(0\ 1)"] & P^{0} \oplus Q^{0} \arrow[r, "0"] \arrow[d, "(0\ 1)"] & 0 \arrow[r] \arrow[d, "0"] & \cdots \\
p(C) = \cdots \arrow[r] & Q^{-2} \arrow[r, "g^{-2}"] \arrow[d, "h^{-2}"] & Q^{-1} \arrow[r, "g^{-1}"] \arrow[d, "h^{-1}"] & Q^{0} \arrow[r, "0"] \arrow[d, "0"] & 0 \arrow[r] \arrow[d, "0"] & \cdots \\
p(A)[1] = \cdots \arrow[r] & P^{-1} \arrow[r, "-f^{-1}"] & P^{0} \arrow[r, "0"] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

Moreover, this can be seen to be in Δ.

Dually, if $\mathcal{A}$ is abelian with enough injectives, then we have the functor $i : \mathcal{A} \to \mathcal{K}(\mathcal{A})$ that maps an object to its injective resolution.

<u>Theorem</u>. Assume that $\mathcal{A}$ is abelian.

(a) If $\mathcal{A}$ has enough projectives, then
$$\mathrm{Ext}^n_{\mathcal{A}}(A,B) = \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(p(A), B[n]).$$

(b) If $\mathcal{A}$ has enough injectives, then
$$\mathrm{Ext}^n_{\mathcal{A}}(A,B) = \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A, i(B)[n]).$$

Proof. We only show (b) since (a) is dual. We have

$$\begin{aligned}
\mathrm{Ext}^n_{\mathcal{A}}(A,B) &= \mathbb{R}^n\left(\mathrm{Hom}_{\mathcal{A}}(A,-)\right)(B) && \left(\begin{smallmatrix}\text{definition}\\ \text{of } \mathrm{Ext}^n(-,B)\end{smallmatrix}\right)\\
&= H^n\left(\mathrm{Hom}_{\mathcal{A}}(A, i(B))\right) && \left(\begin{smallmatrix}\text{definition of}\\ \mathbb{R}^n\end{smallmatrix}\right)\\
&= \mathrm{Hom}_{K(\mathcal{A})}(A, i(B)[n]) && (\text{previous lemma}) \quad \square
\end{aligned}$$

This theorem provides an identification between the long exact sequence obtained from derived functors (Lecture 16) and the long exact sequence coming from a distinguished triangle in $K(\mathcal{A})$ (Lecture 21).

Remark. The above can be generalized in the following way. Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{A}$ and $F: \mathcal{A} \to \mathcal{B}$ be an additive right exact functor between abelian categories with enough projectives. We have a long exact sequence (see Lecture 16)

$$\begin{aligned}
\cdots \to \mathbb{L}_2 F(C) \longrightarrow \mathbb{L}_1 F(A) \longrightarrow \mathbb{L}_1 F(B) \longrightarrow \mathbb{L}_1 F(C) \quad (*)\\
\longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \to 0
\end{aligned}$$

On the other hand, we have a distinguished triangle

$$\rho(A) \longrightarrow \rho(B) \longrightarrow \rho(C) \longrightarrow \rho(A)[1] \text{ in } \mathcal{K}(\mathcal{A}),$$

and the induced functor $F_k : \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{B})$ sends the above to a distinguished triangle

$$F_k\rho(A) \longrightarrow F_k\rho(B) \longrightarrow F_k\rho(C) \longrightarrow F_k\rho(A)[1] \text{ in } \mathcal{K}(\mathcal{B}).$$

Then one can show the existence of a long exact sequence

$$\longrightarrow H^{-2}(F_k\rho(A)) \longrightarrow H^{-1}(F_k\rho(A)) \longrightarrow H^{-1}(F_k\rho(B)) \longrightarrow H^{-1}(F_k\rho(C))$$
$$\longrightarrow H^{0}(F_k\rho(A)) \longrightarrow H^{0}(F_k\rho(B)) \longrightarrow H^{0}(F_k\rho(A)) \longrightarrow \dots \quad (***)$$

Then $(*) = (**)$.

Derived category.

$\mathcal{A}$ — abelian category
$\mathcal{C}(\mathcal{A})$ — category of complexes
$\mathcal{K}(\mathcal{A})$ — the homotopy category of $\mathcal{A}$

Motivation. In homological algebra we care about homology rather than complexes. Hence we want to regard complexes with isomorphic homology, that is quasi isomorphic complexes, as "the same". To do this we make quasi isomorphisms invertible.

<u>Applications</u>. Algebraic geometry, representation theory, topological data analysis and more.

<u>Lemma</u> (Ore condition I) Given a diagram in $K(\mathcal{A})$ of the form

$$\begin{tikzcd} \tilde{A}^\bullet \arrow[r, "f"] \arrow[d, "q"', "\text{qis}"] & B^\bullet \\ A^\bullet & \end{tikzcd} \quad \text{respectively} \quad \begin{tikzcd} & B^\bullet \arrow[d, "r", "\text{qis}"'] \\ A^\bullet \arrow[r, "g"'] & \tilde{B}^\bullet \end{tikzcd}$$

where q respectively r are quasi isomorphisms, we can always complete it to a diagram

$$\begin{tikzcd} \tilde{A}^\bullet \arrow[r, "f"] \arrow[d, "q"', "\text{qis}"] & B^\bullet \arrow[d, "q'", "\text{qis}"'] \\ A^\bullet \arrow[r, "f'"'] & \tilde{B}^\bullet \end{tikzcd} \quad \text{respectively} \quad \begin{tikzcd} \tilde{A}^\bullet \arrow[r, "g'"] \arrow[d, "r'"', "\text{qis}"] & B^\bullet \arrow[d, "r", "\text{qis}"'] \\ A^\bullet \arrow[r, "g"'] & \tilde{B}^\bullet \end{tikzcd}$$

where q' respectively r' are quasi isomorphisms. If moreover f respectively g is a quasi isomorphism, then so is f' respectively g'.

Proof. Consider the distinguished triangles

$$\tilde{A}^\bullet \xrightarrow{q} A^\bullet \xrightarrow{\binom{1}{0}} \mathrm{Cone}(q) \xrightarrow{(0\ 1)} \tilde{A}^\bullet[1] \quad \text{and}$$

$$\mathrm{Cone}(q) \xrightarrow{f[1]\circ(0\ 1)} B^\bullet[1] \longrightarrow \mathrm{Cone}(f[1]\circ(0,1)) \longrightarrow \mathrm{Cone}(q)[1]$$

in $K(\mathcal{A})$. By setting $\widetilde{B^\bullet}[1] := \mathrm{Cone}(f[1]\circ(01))$ and applyin (TR2) on the second triangle, we get a distinguished triangle

$$B^\bullet \xrightarrow{q'} \tilde{B}^\bullet \longrightarrow \mathrm{Cone}(q) \longrightarrow B^\bullet[1]$$

in $K(\mathcal{A})$. Applying (TR3) to the diagram

$$\begin{tikzcd}
\mathrm{Cone}(q)[-1] \arrow[r, "(0\ 1)"] \arrow[d, "\mathrm{id}_{\mathrm{Cone}(q)[-1]}"'] & \tilde{A}^\bullet \arrow[r, "q"] \arrow[d, "f"] & A^\bullet \arrow[r, "\binom{1}{0}"] & \mathrm{Cone}(q) \arrow[d, "\mathrm{id}_{\mathrm{Cone}(q)}"] \\
\mathrm{Cone}(q)[-1] \arrow[r, "f\circ(01)"] & B^\bullet \arrow[r, "q'"] & \tilde{B}^\bullet \arrow[r] & \mathrm{Cone}(q)
\end{tikzcd}$$

we obtain $f' : A^\bullet \longrightarrow \tilde{B}^\bullet$ such that the square

$$\begin{tikzcd}
\tilde{A}^\bullet \arrow[r, "q"] \arrow[d, "f"'] & A^\bullet \arrow[d, "f'"] \\
B^\bullet \arrow[r, "q'"] & \tilde{B}^\bullet
\end{tikzcd}$$

is commutative. It remains to show that q' is a quasi isomorphism. By (TR1)(iii) and (TR 3) we obtain a morphism of triangles

$$\begin{tikzcd}
B^\bullet \arrow[r, "q'"] \arrow[d, "id_{B^\bullet}"] & \tilde{B}^\bullet \arrow[r] \arrow[d, "id_{\tilde{B}^\bullet}"] & \mathrm{Cone}(q) \arrow[r] \arrow[d, "h"] & B^\bullet[1] \arrow[d, "id_{B^\bullet[1]}"] \\
B^\bullet \arrow[r, "q'"] & \tilde{B}^\bullet \arrow[r] & \mathrm{Cone}(q') \arrow[r] & B^\bullet[1]
\end{tikzcd}$$

in $\mathcal{K}(\mathcal{A})$. By the 2 out of 3 property for isomorphisms, we conclude that $\mathrm{Cone}(q) \cong \mathrm{Cone}(q')$. The we have

$$q \text{ quasi isomorphism} \overset{\text{Lecture 13}}{\iff} \mathrm{Cone}(q) \text{ exact}$$
$$\iff \mathrm{Cone}(q') \text{ exact}$$
$$\overset{\text{Lecture 13}}{\iff} q' \text{ quasi isomorphism.}$$

For the last part assume that f is a quasi isomorphism. Then by the definition of $H(-)$ we have

$$H(f') \circ H(q) = H(f' \circ q) = H(q' \circ f) = H(q') \circ H(f).$$

Since f, q, q' are quasi isomorphisms, we have that $H(f)$, $H(q)$ and $H(q')$ are isomorphisms. It follows that $H(f')$ is an isomorphism and so f' is a quasi isomorphism. $\square$