Lecture 4

Equivalences of categories

<u>Def</u>: A functor $F: \mathcal{C} \to \mathcal{D}$ is an <u>equivalence</u> if $\exists G: \mathcal{D} \to \mathcal{C}$ such $F \circ G$ is nat iso to $Id_{\mathcal{D}}$ and $G \circ F$ is nat iso to $Id_{\mathcal{C}}$.

G - <u>quasi inverse</u> of F

<u>Idea</u>: We want to say when to categories are "the same"

Isomorphism of categories is too strong

Equivalence of categories is the correct notion

<u>Theorem</u> $F: \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if it is full, faithful and dense.

<u>Proof</u>: Assume F is an equivalence

$G: \mathcal{D} \to \mathcal{C}$ quasi-inverse.

$\alpha: GF \xrightarrow{\cong} Id_{\mathcal{C}}$ $\beta: FG \xrightarrow{\cong} Id_{\mathcal{D}}$.

nat iso's.

<u>F dense</u>: Let $D \in \mathcal{D}$ Then

$\beta_D: F(G(D)) \xrightarrow{\cong} D$ iso

$\uparrow$ F(object in $\mathcal{C}$)

F faithful: Let $f_1, f_2 : X \to Y$ in $\mathcal{C}$.
Assume $F(f_1) = F(f_2)$. Have commutative squares

$$\begin{array}{ccc} GF(X) & \xrightarrow{GF(f_1)} & GF(Y) \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} \\ X & \xrightarrow[f_1]{} & Y \end{array} \qquad \begin{array}{ccc} GF(X) & \xrightarrow{GF(f_2)} & GF(Y) \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} \\ X & \xrightarrow[f_2]{} & Y \end{array}$$

α_X, α_Y isomorphisms

$$\Rightarrow f_1 = \alpha_Y \circ GF(f_1) \circ \alpha_X^{-1} = \alpha_Y \circ GF(f_2) \circ \alpha_X^{-1} = f_2$$

$(GF(f_1) = GF(f_2))$

F full: $g : F(X) \to F(Y)$ morphism in $\mathcal{D}$.

Consider

$$\begin{array}{ccc} GF(X) & \xrightarrow{G(g)} & GF(Y) \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} \\ X & \xrightarrow{f} & Y \end{array} \qquad (*)$$

$f := \alpha_Y G(g) \alpha_X^{-1}$

Then $F(f), g \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$

Want to show they are equal.

Consider

$$\begin{array}{ccc} GF(X) & \xrightarrow{GF(f)} & GF(Y) \\ \alpha_X \downarrow & & \downarrow \alpha_Y \\ X & \xrightarrow{f} & Y \end{array}$$

commute by naturality $\Rightarrow GF(f) = \alpha_Y^{-1} f \alpha_X$

From $(*)$, $G(g) = \alpha_Y^{-1} f \alpha_X$

$\Rightarrow GF(f) = G(g)$

Now since $G: \mathcal{D} \to \mathcal{C}$ is an equivalence (why?), we know from the argument above that G is faithful

Hence $GF(f) = G(g) \Rightarrow F(f) = g$

so F is full.

<u>Converse</u>: Assume $F: \mathcal{C} \to \mathcal{D}$ is full, faithful and dense.

For each $Z \in \mathcal{D}$, choose (!) $A_Z \in \mathcal{C}$ and an iso $\Theta_Z : F(A_Z) \xrightarrow{\cong} Z$. (F dense)

$\forall Z \xrightarrow{g} W$ in $\mathcal{D}$, consider

$$F(A_Z) \xrightarrow{\Theta_Z} Z \xrightarrow{g} W \xrightarrow{\Theta_W^{-1}} F(A_W)$$

F full & faithful $\Rightarrow \exists$ unique map

$f_g : A_Z \to A_W$ in $\mathcal{C}$ s.t.

$F(f_g) = \Theta_W^{-1} \circ g \circ \Theta_Z$. Define $G: \mathcal{D} \to \mathcal{C}$ by

$G(Z) = A_Z \quad \forall Z \in \mathcal{D}$

$G(g) = G(f_g) \quad \forall$ morphisms g in $\mathcal{D}$.

Then:

- G is a functor (check!)
- $\Theta = (\Theta_Z : FG(Z) \to Z \mid Z \in \mathcal{D}\}$ is a nat iso $F \circ G \to Id_{\mathcal{D}}$ (check!)

$\forall X \in \mathcal{C}$, consider the morphism

$\Theta_{F(X)} : FGF(X) \to F(X)$

F full & faithful $\Rightarrow \exists$ unique morphism

$\xi_X : GF(X) \longrightarrow X$ s.t. $G(\xi_X) = \Theta_{F(X)}$.

Then $\xi = \{\xi_X : GF(X) \to X \mid X \in \mathcal{C}\}$ is a nat iso $GF \xrightarrow{\cong} Id_{\mathcal{C}}$ (check!) ∎

<u>Example</u>: $\mathbb{F}$ field.

$Mat_{\mathbb{F}}$ category:

$Obj\ Mat_{\mathbb{F}} = \mathbb{N} \cup \{0\}$

$Hom_{Mat_{\mathbb{F}}}(m,n) = \{n \times m$ matrices over $\mathbb{F}\}$

composition = matrix multiplication

have natural functor

$$Mat_{\mathbb{F}} \longrightarrow mod\ \mathbb{F}$$

$$n \longrightarrow \mathbb{F}^n$$

This is full, faithful and dense, hence it is an equivalence!

Adjoint functors

Def: $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ functors

(F, G) is an adjoint pair if

$\forall X \in \mathcal{C}$ $Y \in \mathcal{D}$ we have a natural isomorphism

$$\phi_{XY}: \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

Here natural means that for all $f: X' \to X$ in $\mathcal{C}$ & $g: Y \to Y'$ in $\mathcal{D}$

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{D}}(F(X'), Y) & \xrightarrow{\mathrm{Hom}_{\mathcal{D}}(F(f), Y)} & \mathrm{Hom}_{\mathcal{D}}(F(X), Y) & \xrightarrow{\mathrm{Hom}_{\mathcal{D}}(F(X), g)} & \mathrm{Hom}_{\mathcal{D}}(F(X), Y') \\
\downarrow \phi_{X',Y} & & \downarrow \phi_{X,Y} & & \downarrow \phi_{X,Y'} \\
\mathrm{Hom}_{\mathcal{C}}(X', G(Y)) & \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(f, G(Y))} & \mathrm{Hom}_{\mathcal{C}}(X, G(Y)) & \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(X, G(g))} & \mathrm{Hom}_{\mathcal{C}}(X, G(Y'))
\end{array}$$

commutes.

Remark • $\mathcal{C}, \mathcal{D}$ category.

Have category $\mathcal{C} \times \mathcal{D}$ given by

$\mathrm{Obj}(\mathcal{C} \times \mathcal{D}) = \mathrm{Obj}\,\mathcal{C} \times \mathrm{Obj}\,\mathcal{D} = \{(X, Y) \mid X \in \mathcal{C}, Y \in \mathcal{D}\}$

$\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((X,Y),(X',Y'))$

$= \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{D}}(X',Y')$.

Composition defined componentwise.

- $\mathcal{C}$ cat. The association

$(X,Y) \longmapsto \mathrm{Hom}_{\mathcal{C}}(X,Y)$

can be made into a functor

$\mathrm{Hom}_{\mathcal{C}}(-,\cdot): \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Set}$ (check!)

- A pair (F,G), $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ is adjoint

$\iff$ there exists a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(F(-),\cdot) \to \mathrm{Hom}_{\mathcal{C}}(-,G(\cdot))$$

of functors $\mathcal{C}^{op} \times \mathcal{D} \to \mathrm{Set}$.

- (F,G) adjoint, say F <u>left adjoint</u> to G, G <u>right adjoint</u> to F

<u>Exercise</u>: $(X,\leq)$, $(Y,\leq)$ posets.
What is an adjoint pair (F,G) when
$F: \mathcal{C}_{(X,\leq)} \to \mathcal{C}_{(Y,\leq)}$, $G: \mathcal{C}_{(Y,\leq)} \to \mathcal{C}_{(X,\leq)}$?

Example (Free modules) R ring.

Forgetful functor $F: \mathrm{Mod}\,R \to \mathrm{Set}$ has a left adjoint $R^{(-)}: \mathrm{Set} \to \mathrm{Mod}\,R$ given as follows

$$\underset{\text{set}}{X} \longmapsto R^{(X)} = \left\{ \text{maps } f: X \to R \;\middle|\; \begin{matrix} f(x) \neq 0 \text{ for only} \\ \text{finitely many } x \in X \end{matrix} \right\}$$

$\varphi: X \to Y$ maps in Set. Then

$R^{(\varphi)}: R^{(X)} \to R^{(Y)}$ given by

$$R^{(\varphi)}(f): Y \to R$$

$$y \longmapsto \sum_{x \in \varphi^{-1}(y)} f(x)$$

The adjunction iso is

$$\mathrm{Hom}_R(R^{(X)}, M) \longrightarrow \mathrm{Hom}_{\mathrm{Set}}(X, F(M))$$

$$\psi \longmapsto [x \mapsto \psi(\chi_x)]$$

$$[f \mapsto \sum_{x \in X} \underset{\in R}{f(x)} \underset{\in M}{\varphi(x)}] \longleftarrow\!\!| \; \varphi$$

where $\chi_x: X \to R$, $\chi_x(y) = \begin{cases} 0 & x \neq y \\ 1_R & x = y \end{cases}$