where the vertical maps are

$\mathrm{Coker}\, d_A^{n-1} \twoheadrightarrow B^{n+1}(A) \hookrightarrow Z^{n+1}(A)$

and similar for $B^\bullet$, $C^\bullet$.

Thus by Snake lemma get exact sequence

$$\begin{tikzcd} H^n(A^\bullet) \arrow[r, "H^n(f^\bullet)"] & H^n(B^\bullet) \arrow[r, "H^n(g^\bullet)"] & H^n(C^\bullet) \arrow[dll, "\partial"'] \\ H^{n+1}(A^\bullet) \arrow[r, "H^{n+1}(f^\bullet)"] & H^{n+1}(B^\bullet) \arrow[r, "H^{n+1}(g^\bullet)"] & H^{n+1}(C^\bullet) \end{tikzcd}$$

repeating argument $\forall n \in \mathbb{Z}$ proves the claim ∎

Note: $C^\bullet$ complex, then $C^\bullet$ exact $\iff H^n(C^\bullet) = 0 \ \forall n \in \mathbb{Z}$.

Get following corollary

<u>Corollary</u> $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ exact sequence in $C(\mathcal{A})$. If two out of $A^\bullet, B^\bullet, C^\bullet$ are exact, then all of $A^\bullet, B^\bullet, C^\bullet$ are exact.

Lecture 13

Cones and quasi-isomorphisms

We often only care about chain complexes up to its homology.

Def: $f^\bullet: A^\bullet \to B^\bullet$ morphism of complexes.
$f^\bullet$ is a quasi-isomorphism if and only if $H^n(f^\bullet)$ is an isomorphism $\forall n \in \mathbb{Z}$.

How to detect if $f^\bullet$ is a quasi-iso?

- If $f^\bullet$ monomorphism, then get ses of cplxs

$$0 \to A^\bullet \xrightarrow{f^\bullet} B^\bullet \to \mathrm{Coker} f^\bullet \to 0$$

$\rightsquigarrow$ get long exact sequence in homology.
Hence $f^\bullet$ quasi-iso $\iff H^n(f^\bullet)$ iso $\forall n \in \mathbb{Z}$
$\iff H^n(\mathrm{Coker} f^\bullet) \cong 0 \ \forall n \in \mathbb{Z}$
$\iff \mathrm{Coker} f^\bullet$ exact.

What if $f^\bullet$ not mono, can we do something similar?
Yes! Using the cone construction.

Def (Cone): Let $f^\bullet: A^\bullet \to B^\bullet$ morphism in $Ch(\mathcal{A})$.
The cone of $f^\bullet$, denoted $\mathrm{Cone}(f^\bullet)$, is the complex

$$\cdots \xrightarrow{\begin{pmatrix} d_B^{-1} & f^0 \\ 0 & -d_A^0 \end{pmatrix}} \underset{\text{degree } 0}{B^0 \oplus A^1} \xrightarrow{\begin{pmatrix} d_B^0 & f^1 \\ 0 & -d_A^1 \end{pmatrix}} \underset{\text{degree } 1}{B^1 \oplus A^2} \xrightarrow{\begin{pmatrix} d_B^1 & f^2 \\ 0 & -d_A^2 \end{pmatrix}} \cdots$$

Note: $\begin{pmatrix} d_B^i & f^{i+1} \\ 0 & -d_A^{i+1} \end{pmatrix} \circ \begin{pmatrix} d_B^{i-1} & f^i \\ 0 & -d_A^i \end{pmatrix}$

$$= \begin{pmatrix} d_B^i \circ d_B^{i-1} & , & d_B^i \circ f^i - f^{i+1} \circ d_A^i \\ 0 & , & d_A^{i+1} \circ d_A^i \end{pmatrix} = \begin{pmatrix} 0 & \underset{\substack{\text{since } f^\bullet \text{ morphism} \\ \text{of cplxs.}}}{0} \\ 0 & 0 \end{pmatrix}$$

This uses the minus sign in front of $d_A^\bullet$!

Have a canonical monomorphism $B^\bullet \xrightarrow{\binom{1}{0}} \mathrm{Cone}(f^\bullet)$ of complexes. What is its cokernel?

<u>Def</u> (Shift) Let $A^\bullet$ be a complex.
We let $A^\bullet[n]$ denote the obtained by shifting n places to the left, i.e.

$$(A^\bullet[n])^i = A^{i+n} \quad \& \quad d^i_{A^\bullet[n]} = (-1)^n d_{A^\bullet}^{n+i}$$

Note:

- $[n] : Ch(\mathcal{A}) \to Ch(\mathcal{A})$ autoequivalence, with inverse $[-n]$
- $H^i(A^\bullet[n]) = H^{i+n}(A^\bullet)$.

Now let $f : A^\bullet \to B^\bullet$ morphism in $Ch(\mathcal{A})$,
Then have s.e.s of complexes

$$B^\bullet \xrightarrow{\binom{1}{0}} Cone(f) \xrightarrow{(0,1)} A^\bullet[1]$$

In fact, is componentwise split, but not split as a sequence in $Ch(\mathcal{A})$!

Theorem: $f^\bullet : A^\bullet \to B^\bullet$ morphism in $Ch(\mathcal{A})$
The long exact sequence in homology associated to $B^\bullet \xrightarrow{\binom{1}{0}} Cone(f^\bullet) \xrightarrow{(0,1)} A^\bullet[1]$ is.

$$\cdots \to H^n(A^\bullet) \xrightarrow{H^n(f^\bullet)} H^n(B^\bullet) \xrightarrow{H^n\binom{1}{0}} H^n(Cone(f^\bullet)) \xrightarrow{H^n(0,1)} H^{n+1}(A) \xrightarrow{H^{n+1}(f^\bullet)} \cdots$$

Pf: That the second & third map is $H^n\binom{1}{0}$ & $H^n(0,1)$ follows from the theorem on long exact sequences in homology. Only need to show connecting homomorphism ∂ is $H^n(f^\bullet)$. For proof of this see notes ∎

We can now answer our question:

<u>Corollary</u>: $f^\bullet: A^\bullet \to B^\bullet$ morphism in $Ch(\mathcal{A})$. Then $f^\bullet$ quasi-isomorphism $\Leftrightarrow$ $\mathrm{Cone}(f^\bullet)$ exact

<u>Proof</u>: We use the long exact sequence in the previous theorem. Then we get

$f^\bullet$ quasi-iso $\Leftrightarrow$ $H^n(f^\bullet)$ iso $\forall n \in \mathbb{Z}$

$\Leftrightarrow$ $H^n(\mathrm{Cone}(f^\bullet)) \cong 0 \ \forall n \in \mathbb{Z}$ $\Leftrightarrow$ $\mathrm{Cone}(f^\bullet)$ exact ∎

Homotopy category

Want to identify complexes up to quasi-isomorphisms. This is quite complicated to do, so as a first step we consider homotopy equivalences instead.

Let $\mathcal{A}$ be an <u>additive category</u>

<u>Def</u>: • A morphism $f^\bullet: A^\bullet \to B^\bullet$ is <u>null-homotopic</u> if there are morphisms $h^n \in \mathrm{Hom}_{\mathcal{A}}(A^n, B^{n-1})$, $n \in \mathbb{Z}$, s.t. $f^n = d_B^{n-1} \circ h^n + h^{n+1} \circ d_A^n$

$$\begin{tikzcd}
\cdots \arrow[r, "d_A^{-2}"] & A^{-1} \arrow[r, "d_A^{-1}"] \arrow[d, "f^{-1}"] \arrow[dl, dashed, "h^{-1}"'] & A^0 \arrow[r, "d_A^{0}"] \arrow[d, "f^{0}"] \arrow[dl, dashed, "h^{0}"'] & A^1 \arrow[r, "d_A^{1}"] \arrow[d, "f^{1}"] \arrow[dl, dashed, "h^{1}"'] & \cdots \arrow[dl, dashed, "h^{2}"'] \\
\cdots \arrow[r, "d_B^{-2}"'] & B^{-1} \arrow[r, "d_B^{-1}"'] & B^0 \arrow[r, "d_B^{0}"'] & B^1 \arrow[r, "d_B^{1}"'] & \cdots
\end{tikzcd}$$

- Morphisms $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$ are <u>homotopic</u> if $f^\bullet - g^\bullet$ is null-homotopic. In this case we write $f^\bullet \sim g^\bullet$.

- A morphism $f^\bullet : A^\bullet \to B^\bullet$ is a <u>homotopy equivalence</u> if there exists $g^\bullet : B^\bullet \to A^\bullet$ s.t.
$g^\bullet \circ f^\bullet \sim 1_{A^\bullet}$ & $f^\bullet \circ g^\bullet \sim 1_{B^\bullet}$

<u>Note</u>: The following hold (check!)

(1) $\sim$ is an equivalence relation on $\mathrm{Hom}_{C(\mathcal{C})}(A^\bullet, B^\bullet)$

(2) If $f^\bullet \sim g^\bullet$, then $k^\bullet f^\bullet \sim k^\bullet g^\bullet$ & $f^\bullet k^\bullet \sim g^\bullet k^\bullet$ whenever the compositions make sense

(3) If $f_1^\bullet, f_2^\bullet, g_1^\bullet, g_2^\bullet \in \mathrm{Hom}_{Ch(\mathcal{C})}(A^\bullet, B^\bullet)$ and $f_1^\bullet \sim g_1^\bullet$, $f_2^\bullet \sim g_2^\bullet$, then $f_1^\bullet + f_2^\bullet \sim g_1^\bullet + g_2^\bullet$

Def: The <u>homotopy category</u> $K(\mathcal{A})$ is given by

$$\mathrm{Ob}\,K(\mathcal{A}) = \mathrm{Ob}\,Ch(\mathcal{A})$$

$$\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{Ch(\mathcal{A})}(A^\bullet, B^\bullet)/\sim$$

i.e. $f^\bullet$ is equal $g^\bullet$ in $K(\mathcal{A})$ if $f^\bullet \sim g^\bullet$ in $Ch(\mathcal{A})$

<u>Note</u>:

- By (1) & (2) above, we get that $K(\mathcal{A})$ is a category
- By (3) the additive structure on $\mathrm{Hom}_{Ch(\mathcal{A})}(A^\bullet, B^\bullet)$ descends to an additive structure on $\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet)$. Hence $K(\mathcal{A})$ becomes an additive category
- In general $K(\mathcal{A})$ is not an abelian category! Will see later that it is a so-called triangulated category
- There is a canonical functor

  $$Ch(\mathcal{A}) \longrightarrow K(\mathcal{A})$$

  Let $f^\bullet : A^\bullet \to B^\bullet$ morphism in $Ch(\mathcal{A})$. Then

  - $f^\bullet$ is 0 in $K(\mathcal{A})$ iff $f^\bullet$ is null-homotopic in $Ch(\mathcal{A})$

- $f^\bullet$ is a homotopy equivalence in $Ch(\mathcal{A})$ iff $f^\bullet$ is an isomorphism in $K(\mathcal{A})$

Lecture 14

Lemma: $\mathcal{A}$ abelian category. The following hold:

(1) If $f^\bullet : A^\bullet \to B^\bullet$ is nullhomotopic, then $H^n(f^\bullet) = 0 \ \forall n \in \mathbb{Z}$.

(2) H^n descends to a functor $H^n(-): K(\mathcal{A}) \to Ab$

$$\left(\begin{tikzcd} C(\mathcal{A}) \arrow[d] \arrow[dr, "H^n(-)"] & \\ K(\mathcal{A}) \arrow[r, "H^n(-)"'] & Ab \end{tikzcd}\right)$$

(3) If $f^\bullet$ is a homotopy equivalence, then $f^\bullet$ is a quasi-isomorphism.

Proof:

(1) (For Mod R): Have $f^n = d_B^{n-1} h^n + h^{n+1} d_A^n$ for morphisms $h^n : A^n \to B^{n-1}$. Hence for $x \in Z^n(A^\bullet)$, have

$f^n(x) = d_B^{n-1} h^n(x) + h^{n+1} \overset{0}{\overset{\|}{d_A^n(x)}} = d_B^{n-1} h^n(x)$, so

$H^n(f^\bullet)(x + \operatorname{im} d_A^{n-1}) = f^n(x) + \operatorname{im} d_B^{n-1} = d_B^{n-1}(h^n(x)) + \operatorname{im} d_B^{n-1} = 0$