Lecture 21

Some properties of triangulated categories

Throughout, let $(\mathcal{T}, \Sigma, \Delta)$ be a triangulated category.

Lemma. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ be in Δ. Then $g \circ f = 0$, $h \circ g = 0$ and $(\Sigma f) \circ h = 0$.

Proof. By (TR2) it is enough to show $g \circ f = 0$. By (TR1) we have that $X \xrightarrow{id_X} X \xrightarrow{0} 0 \xrightarrow{0} \Sigma X$ is in Δ. Applying (TR3) to

$$\begin{array}{ccccccc}
X & \xrightarrow{id_X} & X & \xrightarrow{0} & 0 & \xrightarrow{0} & \Sigma X \\
\| & & \downarrow{\scriptstyle f} & & & & \| \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X
\end{array}$$

we obtain a map γ such that $g \circ f = \gamma \circ 0 = 0$. $\square$

Theorem. (Long exact Hom-sequence) Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ be in Δ and $T \in \mathcal{T}$. Then the sequences

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{T}}(T, X[n]) \xrightarrow{f[n]\circ -} \mathrm{Hom}_{\mathcal{T}}(T, Y[n]) \xrightarrow{g[n]\circ -} \mathrm{Hom}_{\mathcal{T}}(T, Z[n]) \xrightarrow{h[n]\circ -}$$
$$\hookrightarrow \mathrm{Hom}_{\mathcal{T}}(T, X[n+1]) \xrightarrow{f[n+1]\circ -} \mathrm{Hom}_{\mathcal{T}}(T, Y[n+1]) \xrightarrow{g[n+1]\circ -} \mathrm{Hom}_{\mathcal{T}}(T, Z[n+1]) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{T}}(Z[n], T) \xrightarrow{-\circ (g[n])} \mathrm{Hom}_{\mathcal{T}}(Y[n], T) \xrightarrow{-\circ (f[n])} \mathrm{Hom}_{\mathcal{T}}(X[n], T) \xrightarrow{-\circ (h[n-1])}$$
$$\hookrightarrow \mathrm{Hom}_{\mathcal{T}}(Z[n-1], T) \xrightarrow{-\circ (g[n-1])} \mathrm{Hom}_{\mathcal{T}}(Y[n-1], T) \xrightarrow{-\circ (f[n-1])} \mathrm{Hom}_{\mathcal{T}}(X[n-1], T) \longrightarrow \cdots$$

are exact.

<u>Proof</u>. We prove that the first sequence is exact the other follows dually.
By (TR2) it is enough to show that

$$\mathrm{Hom}_{\mathcal{T}}(T, X) \xrightarrow{f\circ -} \mathrm{Hom}_{\mathcal{T}}(T, Y) \xrightarrow{g\circ -} \mathrm{Hom}_{\mathcal{T}}(T, Z)$$

is exact. Hence we need to show that

$$\mathrm{Im}(f\circ -) = \mathrm{Ker}(g\circ -).$$

•) $\mathrm{Im}(f\circ -) \subseteq \mathrm{Ker}(g\circ -)$: Let $a \in \mathrm{Im}(f\circ -)$. Then $a = f\circ b$ for some $b : T \to X$. Then

$$(g\circ -)(a) = g\circ a = g\circ f\circ b = 0,$$

since $g\circ f = 0$ by previous lemma. Hence $a \in \mathrm{Ker}(g\circ -)$.

•) $\mathrm{Ker}(g\circ -) \subseteq \mathrm{Im}(f\circ -)$: Let $c \in \mathrm{Ker}(g\circ -)$. Then $c : T \to Y$

with $g\circ c=0$. Consider the diagram

$$\begin{array}{ccccccc}
\Sigma^{-1}T & \xrightarrow{0} & 0 & \xrightarrow{0} & T & \xrightarrow{id_T} & T \\
\downarrow{\scriptstyle \Sigma^{-1}c} & & \downarrow{\scriptstyle 0} & & & & \downarrow{\scriptstyle a} \\
\Sigma^{-1}Y & \xrightarrow{-\Sigma^{-1}g} & \Sigma^{-1}Z & \xrightarrow{-\Sigma^{-1}h} & X & \xrightarrow{f} & Y.
\end{array} \qquad (1)$$

The first row in (1) is in Δ by applying right rotation twice to the trivial triangle of T and the second row in (1) is in Δ by appelying right rotation twice to the given triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$.
Since $(-\Sigma^{-1}g)\circ(\Sigma^{-1}c) = -\Sigma^{-1}(g\circ c) = -\Sigma^{-1}0 = 0$, by appelying (TR3) to (1) we obtain $d: T \longrightarrow X$ such that $f\circ d = a$. Hence $a \in Im(f\circ -)$. $\square$

<u>Remark</u>. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ be in Δ.
By the above theorem we have $h\circ g = 0$ and for any morphism $a: T \longrightarrow Z$ such that $h\circ a = 0$, there exists $b: T \longrightarrow Y$ such that $g\circ b = a$:

$$\begin{array}{ccccc}
T & & & & \\
\vdots\,{\scriptstyle \exists b} & \searrow{\scriptstyle a} & & \searrow{\scriptstyle 0} & \\
Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X
\end{array}$$

This is identical to the definition of kernel, except b is not unique. This is sometimes called a <u>weak kernel</u>. A <u>weak cokernel</u> is defined in a similar way. Hence by the above theorem any morphism in a triangle is a weak kernel of the next and a weak cokernel of the previous one.

<u>Theorem</u> (2 out of 3 property for isomorphisms). Let

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \downarrow \Sigma\alpha \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array} \qquad (2)$$

be a morphism of triangles in Δ. If two of the morphisms α, β and γ are isomorphisms, then so is the third one.

<u>Proof</u>. By (TR2) it is enough to show that if α and β are iso, then γ is iso. $Hom_{\mathcal{T}}(-, Z)$ to (2) to obtain

$$
\begin{array}{ccccccccc}
\mathrm{Hom}_{\mathcal{T}}(X,Z) & \xleftarrow{-\circ f} & \mathrm{Hom}_{\mathcal{T}}(Y,Z) & \xleftarrow{-\circ g} & \mathrm{Hom}_{\mathcal{T}}(Z,Z) & \xleftarrow{-\circ h} & \mathrm{Hom}_{\mathcal{T}}(\Sigma X,Z) & \xleftarrow{-\circ \Sigma f} & \mathrm{Hom}_{\mathcal{T}}(\Sigma Y,Z) \\
\uparrow {\scriptstyle -\circ\alpha} & & \uparrow {\scriptstyle -\circ\beta} & & \uparrow {\scriptstyle -\circ\gamma} & & \uparrow {\scriptstyle -\circ\Sigma\alpha} & & \uparrow {\scriptstyle -\circ\Sigma\beta} \\
\mathrm{Hom}_{\mathcal{T}}(X',Z) & \xleftarrow{-\circ f'} & \mathrm{Hom}_{\mathcal{T}}(Y',Z) & \xleftarrow{-\circ g'} & \mathrm{Hom}_{\mathcal{T}}(Z',Z) & \xleftarrow{-\circ h'} & \mathrm{Hom}_{\mathcal{T}}(\Sigma X',Z) & \xleftarrow{-\circ \Sigma f'} & \mathrm{Hom}_{\mathcal{T}}(\Sigma Y',Z)
\end{array}
$$

α, β iso $\Longrightarrow$ $-\circ\alpha, -\circ\beta, -\circ\Sigma\alpha, -\circ\Sigma\beta$ iso by Exercise 1.5.

Long-exact Hom-sequence

$\Longrightarrow$ $-\circ\gamma$ iso by five lemma.

$\Longrightarrow$ $\exists\, \gamma' : Z' \longrightarrow Z$ such that $\gamma'\circ\gamma = \mathrm{id}_Z$

$\Longrightarrow$ γ is split mono.

By applying $\mathrm{Hom}_{\mathcal{T}}(Z',-)$ to (2), we can show that γ is split epi. Then

$\left.\begin{array}{l}\gamma \text{ split mono} \\ \gamma \text{ split epi}\end{array}\right\} \xRightarrow{\text{(exercise)}} \gamma$ isomorphism. $\square$

Triangulated structure of homotopy category

$\mathcal{A}$ – additive category, $\mathcal{C}(\mathcal{A})$ – category of complexes

$\mathcal{K}(\mathcal{A})$ – the homotopy category of $\mathcal{A}$

$[1] : \mathcal{K}(\mathcal{A}) \longrightarrow \mathcal{K}(\mathcal{A})$ – degree shift with inverse $[-1]$

$\Delta :=$ class of all triangles isomorphic to standard triangles $A^\bullet \xrightarrow{f^\bullet} B^\bullet \longrightarrow \mathrm{Cone}(f^\bullet) \longrightarrow A[1]$.

Lemma. Let $A^\bullet \in \mathcal{K}(\mathcal{A})$ be a complex. Assume that $\mathrm{id}_{A^\bullet}$ is null homotopic. Then $A^\bullet \cong 0$ in $\mathcal{K}(\mathcal{A})$.

Proof. End of lecture 13.

Theorem. $(\mathcal{K}(\mathcal{A}), [1], \Delta)$ is a triangulated category.

Proof. (TR2): proposition in Lecture 20 shows closure under left rotation. Closure under right rotation is dual.
(TR1): (i) and (iii) hold by construction. For (ii) we claim that $\mathrm{id}_{\mathrm{Cone}(\mathrm{id}_{A^\bullet})}$ is null homotopic. We have

$$\begin{array}{ccccccccc}
\mathrm{Cone}(\mathrm{id}_{A^\bullet}): & \cdots \longrightarrow & A^{-1}\oplus A^{0} & \xrightarrow{\begin{pmatrix} d^{-1} & 1 \\ 0 & -d^{0}\end{pmatrix}} & A^{0}\oplus A^{1} & \xrightarrow{\begin{pmatrix} d^{0} & 1 \\ 0 & -d^{1}\end{pmatrix}} & A^{1}\oplus A^{2} & \longrightarrow \cdots \\
\downarrow \mathrm{id}_{\mathrm{Cone}(\mathrm{id}_{A^\bullet})} & & \downarrow \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} & \swarrow h=\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix} & \downarrow \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} & \swarrow h=\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix} & \downarrow \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} & \\
\mathrm{Cone}(\mathrm{id}_{A^\bullet}): & \cdots \longrightarrow & A^{-1}\oplus A^{0} & \xrightarrow{\begin{pmatrix} d^{-1} & 1 \\ 0 & -d^{0}\end{pmatrix}} & A^{0}\oplus A^{1} & \xrightarrow{\begin{pmatrix} d^{0} & 1 \\ 0 & -d^{1}\end{pmatrix}} & A^{1}\oplus A^{2} & \longrightarrow \cdots
\end{array}$$

and

$$\begin{pmatrix} d^{-1} & 1 \\ 0 & -d^{0}\end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}\begin{pmatrix} d^{0} & 1 \\ 0 & -d^{1}\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -d^{0} & 0\end{pmatrix} + \begin{pmatrix} 0 & 0 \\ d^{0} & 1\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}.$$

We conclude by the previous lemma that $\mathrm{Cone}(\mathrm{id}_{A^\bullet}) = 0$. Since in $\mathcal{K}(\mathcal{A})$ we have

$$A^\bullet \xrightarrow{\mathrm{id}_{A^\bullet}} A^\bullet \longrightarrow 0 \longrightarrow A^\bullet[1] = A^\bullet \xrightarrow{\mathrm{id}_{A^\bullet}} A^\bullet \longrightarrow \mathrm{Cone}(\mathrm{id}_{A^\bullet}) \longrightarrow A^\bullet[1] \in \Delta,$$

we conclude that (TR1)(ii) holds too.

(TR3): Since we have (TR1), we may assume that triangles are standard triangles. Hence we assume that we have

$$\begin{array}{ccccccc}
A^\bullet & \xrightarrow{f^\bullet} & B^\bullet & \xrightarrow{\binom{1}{0}} & \mathrm{Cone}(f^\bullet) & \xrightarrow{(0\ 1)} & A^\bullet[1] \\
\downarrow{\scriptstyle \alpha^\bullet} & & \downarrow{\scriptstyle \beta^\bullet} & & & & \downarrow{\scriptstyle \alpha^\bullet[1]} \\
A'^\bullet & \xrightarrow{f'^\bullet} & B'^\bullet & \xrightarrow{\binom{1}{0}} & \mathrm{Cone}(f'^\bullet) & \xrightarrow{(0\ 1)} & A'^\bullet[1]
\end{array}$$

such that $f'^\bullet \circ \alpha^\bullet = \beta^\bullet \circ f^\bullet$ in $K(A)$ and we need

$\gamma^\bullet : \mathrm{Cone}(f^\bullet) \longrightarrow \mathrm{Cone}(f'^\bullet)$ such that

$$\gamma^\bullet \circ \binom{1}{0} = \binom{\beta^\bullet}{0} \quad \text{and} \quad (0\ 1)\circ\gamma^\bullet = (0\ \ \alpha^\bullet[1]) \text{ in } K(A)$$

From $f'^\bullet \circ \alpha^\bullet = \beta^\bullet \circ f^\bullet$ in $K(A)$ we obtain

$$\begin{array}{ccccccccc}
A^\bullet: & \cdots \longrightarrow & A^{-1} & \xrightarrow{d_A} & A^0 & \xrightarrow{d_A} & A^1 & \longrightarrow & \cdots \\
{\scriptstyle f'^\bullet\circ\alpha^\bullet}\downarrow\downarrow{\scriptstyle \beta^\bullet\circ f^\bullet} & & {\scriptstyle (f')^{-1}\circ\alpha^{-1}}\downarrow\downarrow{\scriptstyle \beta^{-1}\circ f^{-1}} & \overset{h^0}{\swarrow} & {\scriptstyle (f')^0\circ\alpha^0}\downarrow\downarrow{\scriptstyle \beta^0\circ f^0} & \overset{h^1}{\swarrow} & {\scriptstyle (f')^1\circ\alpha^1}\downarrow\downarrow{\scriptstyle \beta^1\circ f^1} & & \\
B'^\bullet: & \cdots \longrightarrow & B'^{-1} & \xrightarrow{d_{B'}^{-1}} & B'^0 & \xrightarrow{d_{B'}^0} & B'^1 & \longrightarrow & \cdots
\end{array}$$

where

$$\beta^0\circ f^0 - (f')^0\circ\alpha^0 = d_{B'}^{-1}\circ h^0 + h^1\circ d_A^0$$

Define $\gamma^\bullet = \begin{pmatrix} \beta^\bullet & h^{\bullet+1} \\ 0 & \alpha^{\bullet+1}\end{pmatrix}$. Then the diagram

$$\begin{array}{ccccccc}
\mathrm{Cone}(f^\bullet): \cdots \longrightarrow & B^{-1}\oplus A^0 & \xrightarrow{\begin{pmatrix} d_B^{-1} & f^0 \\ 0 & -d_A^0\end{pmatrix}} & B^0\oplus A^1 & \xrightarrow{\begin{pmatrix} d_B^0 & f^1 \\ 0 & -d_A^1\end{pmatrix}} & B^1\oplus A^2 & \longrightarrow \cdots \\
{\scriptstyle \gamma^\bullet}\downarrow & \downarrow{\scriptstyle \begin{pmatrix} \beta^{-1} & h^0 \\ 0 & \alpha^0\end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} \beta^0 & h^1 \\ 0 & \alpha^1\end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} \beta^1 & h^2 \\ 0 & \alpha^2\end{pmatrix}} & \\
\mathrm{Cone}(f'^\bullet): \cdots \longrightarrow & B'^{-1}\oplus A'^0 & \xrightarrow{\begin{pmatrix} d_{B'}^{-1} & f'^0 \\ 0 & -d_{A'}^0\end{pmatrix}} & B'^0\oplus A'^1 & \xrightarrow{\begin{pmatrix} d_{B'}^0 & f'^1 \\ 0 & -d_{A'}^1\end{pmatrix}} & B'^1\oplus A'^2 & \longrightarrow \cdots
\end{array}$$

is a morphism in $K(A)$ and $\gamma^\bullet\circ\binom{1}{0} = \binom{\beta^\bullet}{0}$, $(0\ 1)\circ\gamma^\bullet = (0\ \ \alpha^\bullet[1])$

(exercise).