MA3204 - Exercise 3

1. Let $f\colon R \to S$ be a ring morphism. Show that f induces a faithful functor $f^*\colon \operatorname{Mod} S \to \operatorname{Mod} R$.

2. Recall that a *commutative monoid* is a set X together with an operation $+\colon X \times X \to X$ which is commutative, associative, and has a identity element 0_X. Note that an element x of a monoid X will not necessarily have an inverse $-x$, so X will not necessarily be a group.

   A *pre-semiadditive category* is a category $\mathcal{C}$ together with a monoid structure on each Hom set $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ such that composite

   $$\operatorname{Hom}_{\mathcal{C}}(Y, Z) \times \operatorname{Hom}_{\mathcal{C}}(X, Y) \to \operatorname{Hom}_{\mathcal{C}}(X, Z) \quad (f, g) \mapsto f \circ g$$

   satisfies

   $$f \circ (g_1 + g_2) = f \circ g_1 + f \circ g_2 \quad (f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$$

   and

   $$f \circ 0 = 0 = 0 \circ g.$$

   Similarly to a preadditive category, we can define the biproduct of two objects in a pre-semiadditive category. A pre-semiadditive category is called *semiadditive* if it has a zero object and the biproduct of any two objects exists. Consider the following assertions:

   (i) $\mathcal{C}$ is an semiadditive category;

   (ii) $\mathcal{C}$ is a category that has a zero object $0_{\mathcal{C}}$ and all finite coproducts and products, and such that the canonical map

   $$X_1 \coprod X_2 \coprod \cdots \coprod X_n \to X_1 \prod X_2 \prod \cdots \prod X_n$$

   from a finite coproduct to a finite product is an isomorphism. (here the canonical map is given by the identity map $X_i \xrightarrow{\text{id}} X_i$ for

$1 \leq i \leq n$ and by the map $X_i \to 0_{\mathcal{C}} \to X_j$ factoring through the zero-object when $i \neq j$)

The goal of this exercise is to show that these two statements are equivalent

(a) Assume (ii). Identify the coproduct $X_1 \coprod X_2 \coprod \cdots \coprod X_n$ with the product $X_1 \prod X_2 \prod \cdots \prod X_n$ via the canonical isomorphism and denote it $X_1 \oplus X_2 \oplus \cdots X_n$. Similarly to an additive category, a map $X_1 \oplus X_2 \oplus \cdots X_n \to Y_1 \oplus Y_2 \oplus \cdots \oplus Y_m$ can be written as a $m \times n$-matrix with the (i,j)-entry an element in $\mathrm{Hom}_{\mathcal{C}}(X_j, Y_i)$. Consider the following maps:

- For any X in $\mathcal{C}$, define $\Delta_X := \begin{pmatrix} \mathrm{id}_X \\ \mathrm{id}_X \end{pmatrix} : X \longrightarrow X \oplus X$ such that the projection maps π_1 and π_2 from the product structure satisfy $\pi_1 \Delta_X = \pi_2 \Delta_X = \mathrm{id}_X$;
- For any X in $\mathcal{C}$, define $\nabla_X := \begin{pmatrix} \mathrm{id}_X & \mathrm{id}_X \end{pmatrix} : X \oplus X \longrightarrow X$ such that embedding maps ι_1 and ι_2 from the coproduct structure satisfy $\nabla_X \iota_1 = \nabla_X \iota_2 = \mathrm{id}_X$.
- For any f and g in $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, define
$$f \oplus g := \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} : X \oplus X \longrightarrow Y \oplus Y$$
as the unique map for which $\pi_1(f \oplus g)\iota_1 = f$, $\pi_2(f \oplus g)\iota_2 = g$, $\pi_1(f \oplus g)\iota_2 = 0$ and $\pi_2(f \oplus g)\iota_1 = 0$, where π_1 and π_2 are the projections of $Y \oplus Y$ in the first and second component, respectively, and ι_1 and ι_2 are the embeddings into $X \oplus X$ in the first and second component, respectively.
- For any f and g in $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, define $f + g$ in $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ as the composition $\nabla_Y (f \oplus g) \Delta_X$.

Prove that the operation $+$ defines a structure of a commutative monoid in $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Hint:

- *Let $0 \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ be the unique morphism which factors through the zero object. To show that this is the neutral element under $+$, use that $f \oplus 0$ can be written as a composite*
$$X \oplus X \to X \oplus 0 \to Y \oplus 0 \to Y \oplus Y$$

and use that $X \oplus 0 \cong X$ *and* $Y \oplus 0 \cong Y$.

- *For commutativity of* $+$ *use that we have a commutative diagram*

$$\begin{tikzcd} X \oplus X \arrow[r, "f \oplus g"] \arrow[d, "\tau"] & Y \oplus Y \arrow[d, "\tau"] \\ X \oplus X \arrow[r, "g \oplus f"] & Y \oplus Y \end{tikzcd}$$

where $\tau = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$

- *For associativity of* $+$ *use that the morphisms* $(f+g)+h$ *and* $f+(g+h)$ *are both equal to the composite*

$$X \xrightarrow{\begin{pmatrix} \mathrm{id}_X \\ \mathrm{id}_X \\ \mathrm{id}_X \end{pmatrix}} X \oplus X \oplus X \xrightarrow{\begin{pmatrix} f & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & h \end{pmatrix}} Y \oplus Y \oplus Y \xrightarrow{\begin{pmatrix} \mathrm{id}_Y & \mathrm{id}_Y & \mathrm{id}_Y \end{pmatrix}} Y$$

(b) Prove that (ii)$\Rightarrow$ (i).

Hint: Here you need to show that the addition as defined in (a) makes $\mathcal{C}$ *into a semiadditive category. For the identity*

$$f \circ (g_1 + g_2) = f \circ g_1 + f \circ g_2$$

with $g_1, g_2 \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ *and* $f \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, *use that* $f \circ \nabla_Y = \nabla_Z \circ (f \oplus f)$. *The identity* $(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$ *is proved using a dual argument.*

(c) Finally prove that (i)$\Rightarrow$(ii).

Hint: First show that the isomorphism between the empty product and coproduct just amounts to having a zero object. For $n \geq 2$ *show that the* n*-product as defined in the lecture is both a product and a coproduct.*

In particular, from (ii) we see that being a semiadditive category is a property of $\mathcal{C}$, and not an extra structure. More precisely, we may say that a given category is semiadditive or not, without specifying which monoid structure on the Hom-sets we are considering, since the monoid structure is forced upon us via the construction in (b).

3. Show that a semiadditive category $\mathcal{C}$ is additive if and only if for all objects $X \in \mathcal{C}$ the map

$$\begin{pmatrix} \mathrm{id}_X & \mathrm{id}_X \\ 0 & \mathrm{id}_X \end{pmatrix} : X \oplus X \to X \oplus X$$

is an isomorphism. Conclude that being an additive category is a property of a category $\mathcal{C}$, and not a structure.

Hint: Let $\phi_X =: X \oplus X \to X \oplus X$ *denote the inverse of* $\begin{pmatrix} \mathrm{id}_X & \mathrm{id}_X \\ 0 & \mathrm{id}_X \end{pmatrix}$. *Show that* $\phi = \begin{pmatrix} \mathrm{id}_X & k \\ 0 & \mathrm{id}_X \end{pmatrix}$ *where* k *satisfies* $k + \mathrm{id}_X = 0$. *Hence,* k *is an additive inverse of* id_X *so we can write* $k = -\mathrm{id}_X$. *Finally, show that for a morphism* $f\colon X \to Y$ *the composite* $f \circ (-\mathrm{id}_X)$ *is an additive inverse of* f.

4. Show that $\mathcal{A}$ is an abelian category if and only if $\mathcal{A}^{op}$ is an abelian category.

5. Let $f\colon X \to Y$ be a morphism in an abelian category $\mathcal{A}$. Assume f is both a monomorphism and an epimorphism. Show that f must be an isomorphism (compare with Exercise 1 on Problem sheet 1).

Hint: Use that a monomorphism is a kernel of its cokernel, and that an epimorphism is a cokernel of its kernel.

6. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let (F, G) be an adjoint pair of additive functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$. Show that F is right exact and G is left exact.

Hint: One way to prove this is to use that a left adjoint preserves colimits and a right adjoint preserves limits.

7. Let $\mathcal{A}$ be category and let be $\mathcal{I}$ a small category. Recall that we defined the functor category $\mathrm{Fun}(\mathcal{I}, \mathcal{A})$ in Exercise 7 on the last problem sheet. Show that the following hold:

   - If $\mathcal{A}$ is additive, then $\mathrm{Fun}(\mathcal{I}, \mathcal{A})$ is additive.
   - If $\mathcal{A}$ is abelian, then $\mathrm{Fun}(\mathcal{I}, \mathcal{A})$ is abelian.

- If $\mathcal{A}$ is abelian, then a sequence $F_1 \xrightarrow{\eta} F_2 \xrightarrow{\epsilon} F_3$ in $\mathrm{Fun}(\mathcal{I}, \mathcal{A})$ is exact if and only it is pointwise exact, i.e. $F_1(I) \xrightarrow{\eta_I} F_2(I) \xrightarrow{\epsilon_I} F_3(I)$ is exact in $\mathcal{A}$ for every $I \in \mathcal{I}$.

8. Let $\mathcal{A}$ be an abelian category and let I be the set $\{1, 2\}$ endowed with the partial order $1 \leq 2$.

   (a) Let $\mathrm{Mor}(\mathcal{A})$ denote the category of morphisms of $\mathcal{A}$, i.e., the category whose objects are morphisms of $\mathcal{A}$ and such that, for any two morphisms $f\colon X \longrightarrow Y$ and $g\colon W \longrightarrow Z$, $\mathrm{Hom}_{\mathrm{Mor}(\mathcal{A})}(f, g)$ is the set of all pairs $(h\colon X \longrightarrow W, i\colon Y \longrightarrow Z)$ such that $if = gh$. Show that $\mathrm{Fun}(I, \mathcal{A})$ is equivalent to $\mathrm{Mor}(\mathcal{A})$.

   (b) Show that the kernel and cokernel can be made into functors
   $$\mathsf{Ker}\colon \mathrm{Mor}(\mathcal{A}) \to \mathrm{Mor}(\mathcal{A}) \quad \text{and} \quad \mathsf{Coker}\colon \mathrm{Mor}(\mathcal{A}) \to \mathsf{Mor}(\mathcal{A}).$$

   (c) Show that Ker is right adjoint to Coker. Deduce that Ker is left exact and Coker is right exact.

   (d) Use (c) to show the following: If
   $$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \xrightarrow{g} & X_2 & \xrightarrow{h} & X_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & Y_1 & \xrightarrow{k} & Y_2 & \xrightarrow{l} & Y_3 & \longrightarrow & 0 \end{array}$$
   is a commutative diagram with exact rows, then taking kernels and cokernels we get exact sequences
   $$0 \to \mathrm{Ker}\, f_1 \to \mathrm{Ker}\, f_2 \to \mathrm{Ker}\, f_3$$
   $$\mathrm{Coker}\, f_1 \to \mathrm{Coker}\, f_2 \to \mathrm{Coker}\, f_3 \to 0$$
   in $\mathcal{A}$.

   Hint: Use exercise 6

   (e) Let $\mathcal{A}$ be the category of vectors spaces $\mathsf{Vect}_{\mathbb{K}}$ over a field $\mathbb{K}$ and let F be an object in $\mathrm{Fun}(I, \mathcal{A})$, i.e., F is completely described by two vector spaces $U := F(1)$ and $V := F(2)$ and a linear map $f := F(a_{12})$. Also let $R = T_2(\mathbb{K})$ be the ring of lower triangular 2×2 matrices over $\mathbb{K}$, with addition and multiplication given by addition of matrices and multiplication of matrices. Consider the $\mathbb{K}$-vector space $\Phi(F) := F(1) \oplus F(2)$ and define the following action of the ring R on $\Phi(F)$

$$\mu\colon R \times \Phi(F) \longrightarrow \Phi(F) \qquad \mu(\begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix}, (u, v)) = (\alpha u, \beta f(u) + \gamma v)$$

(i) Show that $(\Phi(F), \mu_{\Phi(F)})$ is a left R-module.

(ii) Show that this defines a functor $\Phi\colon \operatorname{Fun}(I, \mathsf{Vect}_{\mathbb{K}}) \longrightarrow \operatorname{Mod} R^{\mathrm{op}}$ (Recall that $\operatorname{Mod} R^{\mathrm{op}}$ can be identified with the category of left R-modules).

(iii) Show that Φ is an equivalence of categories.

9. Consider two exact sequences in an abelian category $\mathcal{A}$ as follows:

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \qquad 0 \to D \xrightarrow{h} E \xrightarrow{k} F \to 0$$

Show that the following is an exact sequence

$$0 \to A \oplus D \xrightarrow{f \oplus h} B \oplus E \xrightarrow{g \oplus k} C \oplus F \to 0$$

10. Prove the snake lemma in $\operatorname{Mod} R$ using diagram chasing methods (i.e., using elements).