# MA3204 - Exercise 6

1. Let $\mathcal{A}$ be an additive category. Let $f^\bullet : A^\bullet \to B^\bullet$ be a morphism of complexes in $\mathbf{C}(\mathcal{A})$. Show that $\mathrm{Cone}((-1)^n f^\bullet[n]) = \mathrm{Cone}(f^\bullet)[n]$.

2. Let $\mathcal{C}$ be a category. Show that a morphism in $\mathcal{C}$ is an isomorphism if and only if it is both a split monomorphism and a split epimorphism.

3. Let $(\mathcal{T}, \Sigma, \Delta)$ be a triangulated category. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ be a distinguished triangle. Show that the triangles

$$X \xrightarrow{-f} Y \xrightarrow{-g} Z \xrightarrow{h} X[1]$$

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are also distinguished.

4. Let $(\mathcal{T}, \Sigma, \Delta)$ be a triangulated category.

   (a) [Opp16, Exercise VI.1] Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ be a distinguished triangle. If $h = 0$, show that $f$ is a split monomorphism and $g$ is a split epimorphism.
   *Hint: apply (TR3) to a diagram with morphisms between the given triangle and the trivial triangle of $X$.*

   (b) Show that any monomorphism in $\mathcal{T}$ is a split monomorphism and any epimorphism in $\mathcal{T}$ is a split epimorphism. Conclude that a morphism in $\mathcal{T}$ is an isomorphism if and only if it is both a monomorphism and an epimorphism.
   *Hint: use part (a).*

5. Let $(\mathcal{T}, \Sigma, \Delta)$ be a triangulated category. Let $(0, \alpha, 0)$ and $(0, \beta, 0)$ be two endomorphisms of a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$. Show that $\alpha \circ \beta = 0$.
   *Hint: apply (TR3) to a diagram with morphisms between the given triangle and a rotation of the trivial triangle of $Y$.*

6. Let $\mathcal{A}$ be an additive category. The aim of this exercise is to fill-in the details in the proof that $(\mathbf{K}(\mathcal{A}), [1], \Delta)$ satisfies (TR3). Assume that we have a diagram

$$\begin{array}{ccccccc}
A^\bullet & \xrightarrow{f^\bullet} & B^\bullet & \xrightarrow{\binom{1}{0}} & \mathrm{Cone}(f^\bullet) & \xrightarrow{(0\ 1)} & A^\bullet[1] \\
{\scriptstyle \alpha^\bullet}\downarrow & & \downarrow{\scriptstyle \beta^\bullet} & & & & \downarrow{\scriptstyle \alpha^\bullet[1]} \\
A'^\bullet & \xrightarrow{f'^\bullet} & B'^\bullet & \xrightarrow{\binom{1}{0}} & \mathrm{Cone}(f'^\bullet) & \xrightarrow{(0\ 1)} & A'^\bullet[1]
\end{array}$$

   in $\mathbf{K}(\mathcal{A})$ such that $f'^\bullet \circ \alpha^\bullet = \beta^\bullet \circ f^\bullet$. That is, there exist morphisms $h^n : A^n \to B'^{n-1}$ in $\mathcal{A}$ satisfying

$$\beta^n \circ f^n - f'^n \circ \alpha^n = d_{B'}^{n-1} \circ h^n + h^{n+1} \circ d_A^n$$

   for all $n \in \mathbb{Z}$. Define $\gamma^\bullet : \mathrm{Cone}(f^\bullet) \to \mathrm{Cone}(f'^\bullet)$ by $\gamma^n = \begin{pmatrix} \beta^n & h^{n+1} \\ 0 & \alpha^{n+1} \end{pmatrix}$.

   (a) Show that $\gamma^\bullet$ is a morphism in $\mathbf{K}(\mathcal{A})$.

(b) Show that $\gamma^\bullet \circ \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \circ \beta^\bullet$ in $\mathbf{K}(\mathcal{A})$.

(c) Show that $\left(\begin{smallmatrix}0&1\end{smallmatrix}\right) \circ \gamma^\bullet = \alpha^\bullet[1] \circ \left(\begin{smallmatrix}0&1\end{smallmatrix}\right)$ in $\mathbf{K}(\mathcal{A})$.

*Hint: show that the claims (a),(b) and (c) hold in $\mathbf{C}(\mathcal{A})$ even.*

7. Let $\mathcal{A}$ be an additive category. The aim of this exercise is to fill-in the details in the proof that $(\mathbf{K}(\mathcal{A}), [1], \Delta)$ satisfies (TR4). Assume that we have the solid part of the diagram

$$
\begin{array}{ccccccc}
A^\bullet & \xrightarrow{f^\bullet} & B^\bullet & \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} & \operatorname{Cone}(f^\bullet) & \xrightarrow{\left(\begin{smallmatrix}0&1\end{smallmatrix}\right)} & A^\bullet[1] \\
\| & & \downarrow{\scriptstyle g^\bullet} & & \downarrow{\scriptstyle \left(\begin{smallmatrix}g^\bullet&0\\0&1\end{smallmatrix}\right)} & & \| \\
A^\bullet & \xrightarrow{g^\bullet\circ f^\bullet} & C^\bullet & \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} & \operatorname{Cone}(g^\bullet\circ f^\bullet) & \xrightarrow{\left(\begin{smallmatrix}0&1\end{smallmatrix}\right)} & A^\bullet[1] \\
 & & \downarrow{\scriptstyle \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix}1&0\\0&f^\bullet[1]\end{smallmatrix}\right)} & & \downarrow{\scriptstyle f^\bullet[1]} \\
 & & \operatorname{Cone}(g^\bullet) & = & \operatorname{Cone}(g^\bullet) & \xrightarrow{\left(\begin{smallmatrix}0&1\end{smallmatrix}\right)} & B^\bullet[1] \\
 & & \downarrow{\scriptstyle \left(\begin{smallmatrix}0&1\end{smallmatrix}\right)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)} & & \\
 & & B^\bullet[1] & \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} & \operatorname{Cone}(f^\bullet)[1] & &
\end{array}
$$

in $\mathbf{K}(\mathcal{A})$ and consider the dashed part.

(a) Check that the maps in the dashed part are well-defined and that all squares commute in $\mathbf{K}(\mathcal{A})$.

(b) Show that

$$\operatorname{Cone}(f^\bullet) \xrightarrow{\left(\begin{smallmatrix}g^\bullet&0\\0&1\end{smallmatrix}\right)} \operatorname{Cone}(g^\bullet\circ f^\bullet) \xrightarrow{\left(\begin{smallmatrix}1&0\\0&f^\bullet[1]\end{smallmatrix}\right)} \operatorname{Cone}(g^\bullet) \xrightarrow{\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)} \operatorname{Cone}(f^\bullet)[1]$$

is isomorphic to a standard triangle in $\mathbf{K}(\mathcal{A})$.
*Hint: show that the diagram*

$$
\begin{array}{ccccccc}
\operatorname{Cone}(f^\bullet) & \xrightarrow{\left(\begin{smallmatrix}g^\bullet&0\\0&1\end{smallmatrix}\right)} & \operatorname{Cone}(g^\bullet\circ f^\bullet) & \xrightarrow{\left(\begin{smallmatrix}1&0\\0&f^\bullet[1]\end{smallmatrix}\right)} & \operatorname{Cone}(g^\bullet) & \xrightarrow{\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)} & \operatorname{Cone}(f^\bullet)[1] \\
\| & & \| & & {\scriptstyle \left(\begin{smallmatrix}1&0\\0&0\\0&1\\0&0\end{smallmatrix}\right)}\downarrow\ \uparrow{\scriptstyle \left(\begin{smallmatrix}1&0&0&0\\0&f^\bullet[1]&1&0\end{smallmatrix}\right)} & & \| \\
\operatorname{Cone}(f^\bullet) & \xrightarrow{\left(\begin{smallmatrix}g^\bullet&0\\0&1\end{smallmatrix}\right)} & \operatorname{Cone}(g^\bullet\circ f^\bullet) & \xrightarrow{\left(\begin{smallmatrix}1&0\\0&1\\0&0\\0&0\end{smallmatrix}\right)} & \operatorname{Cone}\left(\begin{smallmatrix}g^\bullet&0\\0&1\end{smallmatrix}\right) & \xrightarrow{\left(\begin{smallmatrix}0&0&1&0\\0&0&0&1\end{smallmatrix}\right)} & \operatorname{Cone}(f^\bullet)[1]
\end{array}
$$

*is an isomorphism of triangles in $\mathbf{K}(\mathcal{A})$.*

8. Let $(\mathcal{T}, [1], \Delta)$ be a triangulated category. Let $[1]^{\mathrm{op}} : \mathcal{T}^{\mathrm{op}} \to \mathcal{T}^{\mathrm{op}}$ be the functor defined by

- if $X \in \mathcal{T}^{\mathrm{op}}$, then $X[1]^{\mathrm{op}} = X[-1]$, and
- if $f : X \to Y$ in $\mathcal{T}^{\mathrm{op}}$, so that $f : Y \to X$ in $\mathcal{T}$ and $f[-1] : Y[-1] \to X[-1]$ in $\mathcal{T}$, then $f[1]^{\mathrm{op}}$ is defined as the arrow $f[-1] : X[-1] \to Y[-1]$ in $\mathcal{T}^{\mathrm{op}}$.

Let $\Delta^{\mathrm{op}}$ be a class of triangles in $\mathcal{T}$ defined by

$$\Delta^{\mathrm{op}} = \{Z \xrightarrow{g} Y \xrightarrow{f} X \xrightarrow{-h[1]^{\mathrm{op}}} Z[1]^{\mathrm{op}} \mid X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \in \Delta\}.$$

Show that $(\mathcal{T}^{\mathrm{op}}, [1]^{\mathrm{op}}, \Delta^{\mathrm{op}})$ is a triangulated category.

*Hint: to show that (TR4) holds you may want to use Problem 3.*

# References

[Opp16] Steffen Oppermann. *Notes in homological algebra.* https://folk.ntnu.no/opperman/HomAlg.pdf, 2016.