

## MA3204 - Exercise 4

1. (Closure properties of projectives) Let  $\mathcal{A}$  be an abelian category. Show that the following hold:
  - (a) The zero object in  $\mathcal{A}$  is projective
  - (b) If  $P$  and  $Q$  are projective in  $\mathcal{A}$ , then the biproduct  $P \oplus Q$  is projective in  $\mathcal{A}$ .
  - (c) If  $\{P_i\}_{i \in I}$  is a collection of projective objects in  $\mathcal{A}$ , and if the coproduct  $\coprod_{i \in I} P_i$  exists in  $\mathcal{A}$ , then  $\coprod_{i \in I} P_i$  is projective in  $\mathcal{A}$
  - (d) If  $P$  is projective in  $\mathcal{A}$  and  $P \cong P_1 \oplus P_2$ , then  $P_1$  and  $P_2$  are projective in  $\mathcal{A}$ .
2. Let  $\mathbb{K}$  be a field, and let  $\mathbf{Vect}_{\mathbb{K}}$  be the category of  $\mathbb{K}$ -vector spaces. Show that every object in  $\mathbf{Vect}_{\mathbb{K}}$  is projective and injective.
3. Recall that a left  $R$ -module  $N$  is flat if  $- \otimes_R N$  is an exact functor. Show that the following hold:
  - The flat  $R$ -modules satisfy the closure properties in Problem 1.
  - $R$  is a flat left  $R$ -module.

Conclude that any flat  $R$ -module is projective.

4. Show that  $\mathbb{Q}$  is flat but not projective in  $\mathbf{Ab}$ .
5. Let  $F: \mathcal{A} \rightarrow \mathcal{B}$  be an additive functor between abelian categories and  $G: \mathcal{B} \rightarrow \mathcal{A}$  be a right adjoint to  $F$ . Show that if  $G$  is exact, then  $F(P)$  is projective for any projective object  $P$  in  $\mathcal{A}$ . Dually, show that if  $F$  is exact, then  $G(E)$  is injective for any injective object  $E$  in  $\mathcal{B}$ .

6. Let  $M$  be a right  $R$ -module and let  $N$  be a left  $R$ -module. Show that the canonical morphism  $M \times N \rightarrow M \otimes_R N$  is the universal  $R$ -balanced map with domain  $M \times N$ .
7. Show that
- $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$ , where  $d$  is the greatest common divisor of  $n$  and  $m$ .
  - For any commutative ring  $R$  and any ideals  $I$  and  $J$  of  $R$ ,  $R/I \otimes_R R/J = R/(I + J)$ .
  - For every right  $R$ -module  $M$  over a ring  $R$ , and every left ideal  $I$  of  $R$ ,  $M \otimes_R R/I = M/IM$ .
  - $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \cong 0$ .
  - $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}(i)$ .
8. (The nine lemma) Consider the following diagram in an abelian category  $\mathcal{A}$ .

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A_1 & \xrightarrow{a_1} & A_2 & \xrightarrow{a_2} & A_3 \longrightarrow 0 \\
 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 \\
 0 & \longrightarrow & B_1 & \xrightarrow{b_1} & B_2 & \xrightarrow{b_2} & B_3 \longrightarrow 0 \\
 & & \downarrow g_1 & & \downarrow g_2 & & \downarrow g_3 \\
 0 & \longrightarrow & C_1 & \xrightarrow{c_1} & C_2 & \xrightarrow{c_2} & C_3 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Assume that all the columns are exact. Using what you have learned in the lectures about exact sequences of complexes, show the following:

- If the two upper rows are exact, then the lower row is exact.
- If the two lower rows are exact, then the upper row is exact.
- If the first and third row is exact and  $b_2 \circ b_1 = 0$ , then the middle row is exact.

This result is typically called the nine lemma.

9. Let  $F: \mathcal{A} \rightarrow \mathcal{B}$  be an additive functor between abelian categories. We say that  $F$  *reflects exactness* if whenever

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

is exact, then the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact. Show that if  $F$  is fully faithful and exact, then it reflects exactness.