MA3204 - Exercise 1

1. [1, Exercise I.1].

   - Describe which morphisms in **Set** are monomorphisms, epimorphisms, split monomorphisms, and split epimorphisms.

     Hint: For split monomorphisms $X \to Y$ there are two cases, depending on whether X is the empty set or not.

     To describe the split epimorphisms $X \to Y$ you have to use the axiom of choice

   - (For those who have taken topology courses) Describe which morphisms in **Top** are monomorphisms and which morphism are epimorphisms. Find an example of a morphism that is both a monomorphism and an epimorphism, but not an isomorphism.

   - Let **Ring** denote the category of rings, where the objects are associative and unital rings, and where morphisms are maps preserving the ring structure. Show that in **Ring**, the inclusion $\mathbb{Z} \to \mathbb{Q}$ is both a monomorphism and an epimorphism. This is an example of an epimorphism which is not surjective.

     Hint: To show that $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism, you need to show that any ring morphism $f \colon \mathbb{Q} \to R$ is uniquely determined by its values on the integers $n \in \mathbb{Z}$. Here the identity $f(\frac{m}{n}) = f(m)f(n)^{-1}$ is useful.

2. [1, Exercise I.3] Show that if $f \colon X \to Y$ is an isomorphism, then the $g \colon Y \to X$ satisfying $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$ is uniquely determined (it is denoted by f^{-1})

   Hint: Let $g' \colon Y \to X$ be another map satisfying $g' \circ f = \mathrm{id}_X$ and $f \circ g' = \mathrm{id}_Y$. Consider the composite $g' \circ f \circ g$ and use associativity.

3. [1, Exercise I.2]. Let $\mathcal{C}$ be a category, and let f and g be two composable morphisms. Show that if f and g are monomorphisms, then so is $f \circ g$.

4. [1, Exercise I.3] Show that any split monomorphism is a monomorphism.

5. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor, and let f be a morphism in $\mathcal{C}$. Show that

   - If f is a split monomorphism, then $F(f)$ is a split monomorphism.
   - If f is an isomorphism, then $F(f)$ is an isomorphism.
   - Give an example showing that even if f is a monomorphism, $F(f)$ does not necessarily have to be a monomorphism.

     Hint: Consider the forgetful functor ***Ring*** $\to$ ***Set*** *and the morphism* $\mathbb{Z} \to \mathbb{Q}$ *in exercise 1. What happens when we consider opposite categories?*

Note that Exercise 3, 4, and 5 involved monomorphisms in a category $\mathcal{C}$. To obtain similar results for epimorphisms, one just needs to consider the opposite category $\mathcal{C}^{\mathrm{op}}$ (why?).

References

[1] Steffen Oppermann, 2016 Notes in homological algebra `https://folk.ntnu.no/opperman/HomAlg.pdf`.