MA3204
Homological Algebra
Fall 2018

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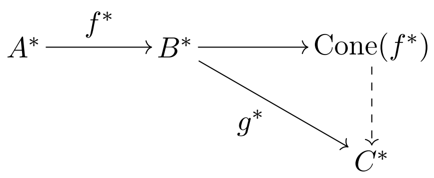
Department of Mathematical Sciences

Exercise set 8

1 Let $A^* \in \mathbf{C}(\mathscr{A})$ be a complex such that $H^i(A^*)=0$ for all $i < 0$. Show that there is a complex B^* such that $B^i = 0$ for all $i < 0$, and a quasi-isomorphism $A^* \to B^*$.
Hint: You can take $B^i = A^i$ for all $i > 0$. The non-trivial choice is that of B^0.

2 Let $f^* : A^* \to B^*$ be a morphism of complexes. Show that $B^* \to \mathrm{Cone}(f^*)$ is a *weak cokernel* of f^* in the homotopy category $\mathbf{K}(\mathscr{A})$, that is, the composition $A^* \xrightarrow{f^*} B^* \to \mathrm{Cone}(f^*)$ is zero, and for any morphism $g^* : B^* \to C^*$ such that $g^* \circ f^* = 0$ in $\mathbf{K}(\mathscr{A})$ there is a (not necessarily unique) factorization



3 Let A^* be a complex. Show that $\mathrm{Cone}(\mathrm{id}_{A^*})$ is split exact.

4

a) Show that a complex P^* is a projective object in $\mathbf{C}(\mathscr{A})$ if and only if it is a split exact complex of projectives.
Hint: For the "only if" direction, consider the exact sequence $\mathrm{Cone}(\mathrm{id}_{P^*}) \to P^*[-1] \to 0$.
For the "if" direction, show that a split exact complex of projectives can be written as a direct sum of exact sequences of the form $\cdots \to 0 \to 0 \to \tilde{P}^n \xrightarrow{\cong} \tilde{\tilde{P}}^{n+1} \to 0 \to 0 \to \cdots$ for some projective objects $\tilde{P}^n$ and $\tilde{\tilde{P}}^{n+1}$. This means that you can reduce the problem to complexes of this form.

b) Show that if $\mathscr{A}$ has enough projectives, then so does $\mathbf{C}(\mathscr{A})$.